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COMPUTING THE CRITICAL FREQUENCIES OF STEPPED SHAFTS

by

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ABSTRACT

Two finite difference methods are presented for representing the critical frequencies of a shaft as the eigenvalues of a matrix. The matrices are well suited for high speed digital computation. The methods are applied to a uniform shaft and compared with the known frequencies.

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I. A simple model of the steady displacement of a rotating shaft is provided by the differential equation

$$(1) \qquad (EIy'')'' = \lambda^2 qy \ , \ EI > 0, \ q > 0$$

with boundary conditions; for example, if the shaft is simply supported, $y(0) = y''(0) = y(L) = y''(L)$ where L is the length of the shaft. EI is the flexural rigidity and q is the linear mass density. The eigenvalues λ_i^2 of this differential problem are known as the critical frequencies of the shaft.

The literature on the subject of numerical approximation of λ^2 seems to be devoted to the adaptation of Mikelstadt's method to high speed computing, with one exception, that being a recent paper [2] on an integral equation approach. Mikelstadt's method is described in [3]. Neither of these methods uses a direct attack on (1). In this paper we apply two well-known and powerful finite difference methods directly to (1), obtaining in both cases linear systems of the form $AX = \mu^2 BX$ where A and B are symmetric, positive definite, and striped. In the first case B is diagonal and μ_i^2 are obtained by the LR method. In the second case we evaluate $\det|A - \mu^2 B|$ by Gaussian elimination and find its zeroes by regula falsi. The first method provides apparent lower bounds for λ^2; the second gives rigorous upper bounds.

II. The first method is based on a continuity principle. To simplify the notation let $p = EI$, so that

$$(2) \qquad (py'')'' = \lambda^2 qy$$

We suppose that p and q are step functions with common points of discontinuity. If we now assume that solutions of (2) are integrable, then by successive integrations it can be shown that the following functions are continuous

$$h^1 = (py'')'$$

$$h^2 = py''$$

$$h^3 = y'$$

$$h^4 = y$$

It must be understood that these continuity conditions are physical ones. If they do not hold it means that (1) is not the proper model.

We set up a mesh

$$x_k = x_{k-1} + \Delta_{k-1/2} \qquad k = 0, 1, \cdots, N$$

$$x_o = 0$$

in such a way that every discontinuity of p and q is some x_k. Set

$$p_{k+1/2} = p(x_{k+1/2}) \text{ , etc.}$$

The derivation of the difference equations is started by setting

$$y_k = \frac{2}{\Delta_{k+1/2} + \Delta_{k-1/2}} \int_{x_{k-1/2}}^{x_{k+1/2}} \frac{(py'')''}{q} \, dx$$

$$= \frac{2}{\Delta_{k+1/2} + \Delta_{k-1/2}} \left(\frac{h_k^1 - h_{k-1/2}^1}{q_{k-1/2}} + \frac{h_{k+1/2}^1 - h_k^1}{q_{k+1/2}} \right)$$

If we approximate the continuity of y by the relation

$$\frac{1}{\Delta_{k-1/2}} \int_{x_{k-1/2}}^{x_k} \frac{(py'')''}{q} \, dx = \frac{1}{\Delta_{k+1/2}} \int_{x_k}^{x_{k+1/2}} \frac{(py'')''}{q} \, dx$$

then we can eliminate h_k^1 to obtain

$$\underline{\mu}^2 y_k = \frac{1}{\bar{q}_k} (h^1_{k+1/2} - h^1_{k-1/2})$$

where

$$\bar{q}_k = \frac{q_{k+1/2} \Delta_{k+1/2} + q_{k-1/2} \Delta_{k-1/2}}{2}$$

Since all functions are smooth at $k+1/2$, put

$$h^1_{k+1/2} = \frac{h^2_{k+1} - h^2_k}{\Delta_{k+1/2}}$$

Proceeding as before, we write

$$h^2_k = \frac{2}{\Delta_{k-1/2} + \Delta_{k+1/2}} \int_{x_{k-1/2}}^{x_{k+1/2}} py'' \, dx$$

and

$$\frac{1}{\Delta_{k-1/2}} \int_{x_{k-1/2}}^{x_k} py'' \, dx = \frac{1}{\Delta_{k+1/2}} \int_{x_k}^{x_{k+1/2}} py'' \, dx$$

to get

$$h_k = \bar{p}_k(h^3_{k+1/2} - h^3_{k-1/2})$$

where

$$\bar{p}_k = \frac{2p_{k-1/2}\, p_{k+1/2}}{p_{k-1/2}\, \Delta_{k+1/2} + p_{k+1/2}\, \Delta_{k-1/2}}$$

Finally

$$h^3_{k+1/2} = \frac{y_{k+1} - y_k}{\Delta_{k+1/2}}$$

The resultant difference equation is

(2a) $$\underline{\mu}^2\, \bar{q}_k y_k = \frac{1}{\Delta_{k+1/2}}\left[\bar{p}_{k+1}\left(\frac{y_{k+2} - y_{k+1}}{\Delta_{k+3/2}} - \frac{y_{k+1} - y_k}{\Delta_{k+1/2}}\right)\right.$$

$$\left. - \bar{p}_k\left(\frac{y_{k+1} - y_k}{\Delta_{k+1/2}} - \frac{y_k - y_{k-1}}{\Delta_{k-1/2}}\right)\right]$$

$$- \frac{1}{\Delta_{k-1/2}}\left[\bar{p}_k\left(\frac{y_{k+1} - y_k}{\Delta_{k+1/2}} - \frac{y_k - y_{k-1}}{\Delta_{k-1/2}}\right)\right.$$

$$\left. - \bar{p}_{k-1}\left(\frac{y_k - y_{k-1}}{\Delta_{k-1/2}} - \frac{y_{k-1} - y_{k-2}}{\Delta_{k-3/2}}\right)\right]$$

with boundary conditions

$$y_o = y_N = 0$$

$$p_o = p_N = 0$$

In matrix form we have

(3) $$AX = \underline{\mu}^2 BX$$

where

$$A = \{a_{ij}\} , \qquad B = \{b_{ij}\}$$

and

$$a_{ij} = a_{ji}$$

$$a_{ij} = 0 \text{ if } |i - j| > 2$$

$$a_{ii} = \frac{1}{\Delta_{i+1/2}}\left[\bar{p}_{i+1} \frac{1}{\Delta_{i+1/2}} + \bar{p}_i\left(\frac{1}{\Delta_{i+1/2}} + \frac{1}{\Delta_{i-1/2}}\right)\right]$$

$$- \frac{1}{\Delta_{i-1/2}}\left[\bar{p}_i\left(\frac{1}{\Delta_{i+1/2}} + \frac{1}{\Delta_{i-1/2}}\right) + \bar{p}_{i+1}\frac{1}{\Delta_{i-1/2}}\right]$$

$$a_{i,i+1} = - \frac{1}{\Delta_{i+1/2}}\left[\bar{p}_{i+1}\left(\frac{1}{\Delta_{i+3/2}} + \frac{1}{\Delta_{i+1/2}}\right) + \bar{p}_i\frac{1}{\Delta_{i+1/2}}\right]$$

$$+ \frac{1}{\Delta_{i-1/2}}\bar{p}_i\frac{1}{\Delta_{i+1/2}}$$

$$a_{i,i+2} = \frac{1}{\Delta_{i+1/2}}\bar{p}_{i+1}\frac{1}{\Delta_{i+3/2}}$$

$$b_{ij} = 0 \,, \qquad i \neq j$$

$$b_{ii} = (\bar{q}_i)$$

III. The second method is based on a variational principle. Let

$$\mu^2(y) = \frac{\int_0^L p(y'')^2 \, dx}{\int_0^L qy^2 \, dx} \tag{4}$$

Let Z be the space of all continuously differentiable functions on L with piecewise continuous second derivatives which also satisfy the boundary conditions. Let S_m be any m-dimensional subspace of Z. Let $\overline{\mu}_i^2$, i = 1, 2, $\cdots$, m be the stationary value of $\mu^2(y)$ as y varies over S_m. Then it is known that

$$\lambda_i^2 \leq \overline{\mu}_i^2, \qquad i = 1, 2, \cdots, m$$

We choose as the subspace $C_{2(N-1)}$, which we define as the set of all continuous functions having continuous first derivatives, and which are cubic polynomials in the intervals (x_k , x_{k+1}), k = 0, 1, $\cdots$, N - 1. Each function in $C_{2(N-1)}$ is determined by the value of it and its first derivative at all the interior mesh points. Call these values y_i, and y_i', i = 1, 2, $\cdots$, N - 1. Then for any y(x) in $C_{2(N-1)}$

$$y(x) = y(x , y_1 , \cdots , y_1' , \cdots)$$

The conditions for stationarity of $\mu^2(y)$ are

$$\frac{\partial(\mu^2(y))}{\partial y_i} = 0$$

$$\frac{\partial\mu^2(y)}{\partial y_i'} = 0 , \qquad i = 1, 2, \cdots, N - 1$$

or

$$\int_0^L y^2 q \, dx \frac{\partial}{\partial y_i} \int_0^L p(y'')^2 \, dx = \int_0^L p(y'')^2 \, dx \frac{\partial}{\partial y_i} \int_0^L y^2 q \, dx$$

and

$$\int_0^L y^2 q \, dx \frac{\partial}{\partial y_i'} \int_0^L p(y'')^2 \, dx = \int_0^L p(y'')^2 \, dx \frac{\partial}{\partial y_i'} \int_0^L y^2 q \, dx$$

Using (4) we have

(5)
$$\frac{\partial}{\partial y_i} \int_0^L p(y'')^2 \, dx = \bar{\mu}^2 \frac{\partial}{\partial y_i} \int_0^L y^2 q \, dx$$

$$\frac{\partial}{\partial y_i'} \int_0^L p(y'')^2 \, dx = \bar{\mu}^2 \frac{\partial}{\partial y_i'} \int_0^L y^2 q \, dx$$

so that again

(6) $AX = \bar{\mu}^2 BX$

where A and B are of necessity positive definite and can be chosen symmetric. The calculation of A and B is extremely tedious and will be omitted. We will give the equations for the cubics in order to aid the reader who wishes to use this method with other boundary conditions. We have

$$y(x) = ax^3 + bx^2 + cx + d$$

where

a) For $0 < i < i + 1 < N$, $x_i \leq x \leq x_{i+1}$

$$a = \frac{\Delta_{i+1/2}(y'_i + y'_{i+1}) - 2(y_{i+1} - y_i)}{(\Delta_{i+1/2})^3}$$

$$b = \frac{3(y_{i+1} - y_i) - \Delta_{i+1/2}(2y'_i + y'_{i+1})}{(\Delta_{i+1/2})^2}$$

$$c = y'_i$$

$$d = y_i$$

b) In the simply supported case, for $0 \leq x \leq x_1$

$$a = \frac{y_i' \Delta_{1/2} - y_1}{2(\Delta_{1/2})^3}$$

$$b = 0$$

$$c = \frac{3y_i - y_i' \Delta_{1/2}}{2\Delta_{1/2}}$$

$$d = 0$$

and for $x_{N-1} \leq x \leq x_N$

$$a = \frac{y_{N-1}' \Delta_{N-1/2} + y_{N-1}}{2(\Delta_{N-1/2})^3}$$

$$c = - \frac{3y_{N-1} + \Delta_{N-1/2} y_{N-1}'}{2\Delta_{N-1/2}}$$

$$b = d = 0$$

If the vector x is taken to be

$$x = (y_1, y_1', \cdots, y_{N-1}, y_{N-1}')$$

then the matrices A and B are block tridiagonal, as follows:

$$A = \begin{pmatrix} \beta_1 & \gamma_1 & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & O \\ & \ddots & \ddots & \ddots & \\ O & & & \alpha_{N-1} & \beta_{N-1} \end{pmatrix} .$$

$$B = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & O \\ & \ddots & \ddots & \ddots & \\ O & & & a_{N-1} & b_{N-1} \end{pmatrix} .$$

$$\alpha_i = \begin{pmatrix} \dfrac{-24p_{i-1/2}}{\Delta^3_{i-1/2}} & \dfrac{-12p_{i-1/2}}{\Delta^2_{i-1/2}} \\ & \\ \dfrac{12p_{i-1/2}}{\Delta^2_{i-1/2}} & \dfrac{4p_{i-1/2}}{\Delta_{i-1/2}} \end{pmatrix} .$$

$$\beta_i = \begin{pmatrix} \dfrac{24p_{i+1/2}}{\Delta^3_{i+1/2}} + \dfrac{24p_{i-1/2}}{\Delta^3_{i-1/2}} & \dfrac{12p_{i+1/2}}{\Delta^3_{i+1/2}} - \dfrac{12p_{i-1/2}}{\Delta^2_{i-1/2}} \\ \\ \dfrac{12p_{i+1/2}}{\Delta^2_{i+1/2}} \quad \dfrac{12p_{i-1/2}}{\Delta^2_{i-1/2}} & \dfrac{8p_{i+1/2}}{\Delta_{i+1/2}} \quad \dfrac{8p_{i-1/2}}{\Delta_{i-1/2}} \end{pmatrix}$$

$$\gamma_i = \begin{pmatrix} \dfrac{-24p_{i+1/2}}{\Delta^3_{i+1/2}} & \dfrac{12p_{i-1/2}}{\Delta^2_{i+1/2}} \\ \\ \dfrac{-12p_{i+1/2}}{\Delta^2_{i+1/2}} & \dfrac{4p_{i+1/2}}{\Delta_{i+1/2}} \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} \dfrac{24p_{3/2}}{\Delta^3_{3/2}} + \dfrac{6p_{1/2}}{\Delta^3_{1/2}} & \dfrac{12p_{3/2}}{\Delta^2_{3/2}} - \dfrac{6p_{1/2}}{\Delta^2_{1/2}} \\ \\ \dfrac{12p_{3/2}}{\Delta^2_{3/2}} + \dfrac{6p_{1/2}}{\Delta^2_{1/2}} & \dfrac{8p_{3/2}}{\Delta_{3/2}} + \dfrac{6p_{1/2}}{\Delta_{1/2}} \end{pmatrix}$$

$$\beta_{N-1} = \begin{pmatrix} \dfrac{24p_{N-3/2}}{\Delta^3_{N-3/2}} + \dfrac{6p_{N-1/2}}{\Delta^3_{N-1/2}} & \dfrac{-12p_{N-3/2}}{\Delta^2_{N-3/2}} + \dfrac{6p_{N-1/2}}{\Delta^2_{N-1/2}} \\ \\ \dfrac{-12p_{N-3/2}}{\Delta^2_{N-3/2}} + \dfrac{6p_{N-1/2}}{\Delta_{N-1/2}} & \dfrac{8p_{N-3/2}}{\Delta_{N-3/2}} + \dfrac{6p_{N-1/2}}{\Delta_{N-1/2}} \end{pmatrix}$$

$$a_i = \begin{pmatrix} \dfrac{9}{35}\, q\Delta_{i-1/2} & \dfrac{13}{210}\, q\Delta^2_{i-1/2} \\ \\ \dfrac{-13}{210}\, q\Delta^2_{i-1/2} & \dfrac{-1}{70}\, q\Delta^3_{i-1/2} \end{pmatrix}$$

$$c_i = \begin{pmatrix} \dfrac{9}{35}\, q\Delta_{i+1/2} & \dfrac{-13}{210}\, q\Delta^2_{i+1/2} \\ \\ \dfrac{13}{210}\, q\Delta^2_{i+1/2} & \dfrac{-1}{70}\, q\Delta^3_{i+1/2} \end{pmatrix}$$

$$b_i = \begin{pmatrix} \frac{26}{35}\left(q\Delta_{i+1/2} + q\Delta_{i-1/2}\right) & \frac{11}{105}\left(q\Delta^2_{i+1/2} - q\Delta^2_{i-1/2}\right) \\ \\ \frac{11}{105}\left(q\Delta^2_{i+1/2} - q\Delta^2_{i-1/2}\right) & \frac{2}{105}\left(q\Delta^3_{i+1/2} + q\Delta^3_{i-1/2}\right) \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \frac{26}{35}\, q\Delta_{3/2} + \frac{34}{35}\, q\Delta_{1/2} & \frac{11}{105}\, q\Delta^2_{3/2} - \frac{6}{35}\, q\Delta^2_{1/2} \\ \\ \frac{11}{105}\, q\Delta^2_{3/2} - \frac{6}{35}\, q\Delta^2_{1/2} & \frac{2}{105}\, q\Delta^3_{3/2} + \frac{4}{105}\, q\Delta^3_{1/2} \end{pmatrix}$$

$$b_{N-1} = \begin{pmatrix} \frac{26}{35}\, q\Delta_{N-3/2} + \frac{34}{35}\, q\Delta_{N-1/2} & \frac{6}{35}\, q\Delta^2_{N-1/2} - \frac{11}{105}\, q\Delta^2_{N-3/2} \\ \\ \frac{6}{35}\, q\Delta^2_{N-1/2} - \frac{11}{105}\, q\Delta^2_{N-3/2} & \frac{2}{105}\, q\Delta^3_{N-3/2} + \frac{4}{105}\, q\Delta^3_{N-1/2} \end{pmatrix}$$

IV. To solve (3) for $\underline{\mu}^2$ we use the SS^T method [1]. First, since B is diagonal with positive diagonal elements we can write

$$B^{-1}AX = \underline{\mu}^2 X$$

The matrices A and B are positive definite and symmetric, but $B^{-1}A$ is not symmetric. It can be symmetrized by setting

$$C = [b_{ii}^{-1/2}]$$

so that $B^{-1} = C^2$

Then if

$$D = CAC$$

we have

$$CDC^{-1} = C^2 A = B^{-1}A$$

so that D is similar to $B^{-1}A$ and has the same eigenvalues. D is symmetric and positive definite. Any such matrix D has a square root; that is, there exists a matrix S, lower triangular, such that

$$D = SS^T$$

If D is striped so is S. The SS^T algorithm is the following:

a) Let $D^{(0)} = D$

b) Given $D^{(k)}$, find $S^{(k)}$ such that

$$D^{(k)} = S^{(k)} S^{(k)T}$$

c) Let $D^{(k+1)} = S^{(k)T} S^{(k)}$

The sequence of matrices $D^{(k)}$ converges to a diagonal matrix with the eigenvalue of $D^{(0)}$ on the diagonal. The off-diagonal elements of the last row and column converge to zero first, at which time they can be discarded and the process continued with successively smaller matrices. For our particular $D^{(0)}$ the smallest eigenvalue appears first, then the next smallest, etc.

The matrix S is found by the following recursion. If $D = (D_{ij})$

$$S_{jj} = \sqrt{D_{jj} - \sum_{r=1}^{j-1} S_{jr}^2}$$

$$S_{ij} = \left(D_{ij} - \sum_{r=1}^{j-1} S_{ir} S_{jr} \right) \left(S_{jj} \right)^{-1} \qquad i = j + 1,\ j + 2,\ \cdots$$

$j = 1, 2, \cdots$

If we now consider the variational method we find that the matrices occurring there do not lend themselves to the above method, the reason being that now B is not diagonal so $B^{-1}A$ is not striped. It turns out to be quite feasible with the Stretch computer to find the zeroes of $\det|A - \tau B|$, with A and B defined in part III, by direct computation. So, let

$$f(\tau) = \det|A - \tau B|$$

For any τ the determinant is evaluated by Gaussian elimination without pivoting, but programmed to take advantage of the sparseness of $A - \tau B$. The zeroes of $f(\tau)$ are found by regula falsi; that is, the iteration

$$\tau^{(k+1)} = \tau^{(k)} - f(\tau^k) \frac{\tau^k - \tau^{k-1}}{f(\tau^k) - f(\tau^{k-1})}$$

To find $\overline{\mu}_i^2$ we start with

$$\tau^o = \underline{\mu}_i^2$$

$$\tau^1 = \underline{\mu}_i^2(1 + \epsilon)$$

for some small ϵ. The $\underline{\mu}_i^2$ are the solutions of (3).

We are able to report on only one actual calculation with the above method. We took

$$p = q \equiv 1$$

$$L = 10$$

We used 20 intervals with the continuity method and 10 intervals with the variational method and obtained the following values:

$10^2 \frac{\underline{\mu}}{2\pi}$	$10^2 \frac{\lambda}{2\pi}$	$10^2 \frac{\overline{\mu}}{2\pi}$
1.56757	1.570799	1.57081
6.23168	6.2832	6.28387
13.877	14.137	14.145

For this problem it is easy to show that $\underline{\mu} < \lambda$; however, this is not known in general. We have observed that for some problems with discontinuous p and q the $\underline{\mu}$'s seem to increase with decreasing Δx.

It is clear from the table that the upper bounds $\overline{\mu}$ are more accurate than the $\underline{\mu}$; however, this comparison is not completely fair since the matrices determining $\overline{\mu}$ have seven non-zero diagonals, while those determining $\underline{\mu}$ have five. The variational method should be compared with a seven point form of the continuity method, but we have not been able to work this form out in the general case.

V. We should like to conclude with a proof that A in (3) is positive definite.

For any vector $w = (y_1 , \cdots , y_{N-1})$ we have to show that $(Aw,w) > 0$ if $w \not\equiv 0$, where

$$(v,w) = \sum_{i=1}^{N-1} v_i w_i$$

Define the following operators

$$(Eu)_i = u_{i+1/2}$$

$$(\hat{\Delta}u)_i = (E - E^{-1})u = u_{i+1/2} - u_{i-1/2}$$

The operator E is unitary on the set of u's which vanish outside some interval, which we may represent by $u_k = 0$ if $k \leq 0$ or $k \geq N$. The unitarity means

$$E^* = E^{-1} , \qquad \text{where } E^* = \text{adjoint of } E$$

Therefore

$$(\hat{\Delta}u,v) = -(u,\hat{\Delta}v)$$

Let

$$h_{i+1/2} = \frac{1}{\Delta_{i+1/2}}$$

An examination of (2a) shows that

$$Aw = (\hat{\Delta} h \hat{\Delta} \bar{p} \hat{\Delta} h \hat{\Delta}) w$$

Putting $R = \hat{\Delta} h \hat{\Delta}$

we have

$$R^* = R$$

Therefore

$$(Aw, w) = (R\bar{p}Rw, w) = (\bar{p}Rw, Rw)$$

$$= \sum \bar{p}_i (Rw)_i^2 \geq 0$$

Equality can occur only if $Rw = 0$, which means $h\hat{\Delta}w$ is constant; however, since $w_o = w_N = 0$, this implies $w \equiv 0$.

Bibliography

1. Rutishauser, H., "Solution of Eigenvalue Problems with LR transformation," in NBS Appl. Math. Series, No. 49, 1958, p. 47.

2. Stern, M. and Stippes, M., A unified numerical treatment for vibrating beams, J. Soc. Ind. Appl. Math, v. 11, No. 2, 413 (1963).

3. Timoshenko, S., and D. H. Young, Vibration Problems in Engineering, Third Ed., D. Van Nostrand Company, Inc. 1955.