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TITLE: GROUP-INVARIANT SOLUTIONS OF HYDRODYNAMICS

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SUBMITTED TO: Internal Report

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FORM NO. 836 R4
ST. NO. 2629 5/81

GROUP-INVARIANT SOLUTIONS OF HYDRODYNAMICS

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1. Introduction

The equations of hydrodynamics, being nonlinear, are in general difficult to solve analytically. A great deal of effort has therefore gone into the numerical solution of these equations, using a wide variety of algorithms. Issues associated with these numerical solutions include accuracy and stability of the algorithms and their associated solutions. Comparison to experiment is one basic way to address the validity of numerical solutions, but the issues of diagnostics and experimental error are always present. Further, there are regimes for which experimental results are either costly or impossible to obtain. Due to this, analytic solutions to such equations in relevant physical regimes have been sought. Such analytic, exact solutions can be used for three purposes: (1) benchmarks for numerical algorithms, (2) the basis for analytic models, (3) to provide insight into more general solutions.

One method of constructing analytic solutions that has been highly successful is the use of Lie group reductions. This technique was invented by Sophus Lie in the latter part of the 19th Century specifically to find solutions to differential equations, both ordinary and partial. Since their inception, Lie groups have been recognized as a powerful formalism applicable to a wide variety of physical and mathematical problems. Ironically, their utility as originally intended as tools to solve differential equations fell out of use until the middle of this century. In the past few decades there has been an explosion in the application of Lie groups to differential equations.[1–6]

The bottom line of the technique is this: invariance of a differential equation under the action of a Lie group allows (1) the reduction in order of ordinary differential equations (ODE's), (2) the reduction of the number of independent variables of partial differential equations (PDE's), and (3) construction of new solutions from existing ones. Each allowed nontrivial invariance group provides such a reduction. Further, chains of reduction are generally possible such that invariance of a differential equation under an n-parameter solvable Lie group allows n such reductions, taking and $n-$th order ODE into quadrature and a PDE with $n+1$ independent variables into an ODE. Systems of equations are reduced in the same fashion.

The first step in this procedure is to identify the Lie groups of transformations that leave a differential equation invariant, that is, the continuous transformations of the space of dependent and independent variables into new variables such that the differential equation written in terms of the new variables is identical to the old differential equation. Once these transformations with this property are found, two things can be done. New solutions can be found as transformations of old solutions, since the transformations take the solution surface into itself (the differential equation is invariant). Second, as a consequence of this invariance, new coordinates can be identified in terms of which the differential equation takes a simpler, reduced form. It is usually easier to obtain a solution in this reduced space, which can then be translated back into the original space to provide a solution to the differential equation.

Starting with PDE's, one attempts to do reductions until ODE's are reached. Solutions of these ODE's are sought and then transformed back into the original variables where they are solutions of the original PDE's. These group-invariant solutions are only particular solutions to the PDE's, not general solutions. They exist and evolve only for particular initial/boundary conditions. Often the special initial/boundary conditions are physically relevant and the particular solutions are physically interesting. Some solutions are more artificial. All the particular solutions can be used as numerical benchmarks, although issues of stability should be considered.

2. Lie Groups Applied to Differential Equations

A Lie group is a collection of elements along with a binary operation such that the four group axioms, closure, associativity, existence of an identity and inverse are satisfied. A point transformation is a transformation of a point in space to a new point according to some defining relations, such as

$$\tilde{x} = f_1(x,y), \quad \tilde{y} = f_2(x,y).$$

(They are called point transformations because the transformation functions f_i depend only on the point (x,y) values and not on derivatives.) If we parameterize this transformation with a continuous parameter a such that the transformation functions f_1 and f_2 are continuous and continuously differentiable to all orders in x, y, and a, we have a collection of continuous point transformations

$$\tilde{x} = f_1(x,y;a), \quad \tilde{y} = f_2(x,y;a).$$

We can now consider the collection of all these transformations for all allowed values of the parameter a. We identify first an identity element a_0 that takes all points back into themselves:

$$x = f_1(x,y;a_0), \quad y = f_2(x,y;a_0).$$

Further, we consider a binary operation among this collection of transformations: a combined transformation that does the action of two consecutive transformations. That is, let the parameter a take the point (x,y) into $(\tilde{x},\tilde{y})$ and the parameter b take the point $(\tilde{x},\tilde{y})$ into another point $(\tilde{\tilde{x}},\tilde{\tilde{y}})$:

$$\tilde{x} = f_1(x,y;a), \quad \tilde{y} = f_2(x,y;a), \qquad \tilde{\tilde{x}} = f_1(\tilde{x},\tilde{y};b), \quad \tilde{\tilde{y}} = f_2(\tilde{x},\tilde{y};b).$$

A binary operation would then be the transformation with parameter c that takes the point (x,y) directly into the point $(\tilde{\tilde{x}},\tilde{\tilde{y}})$:

$$\tilde{\tilde{x}} = f_1(x,y;c), \quad \tilde{\tilde{y}} = f_2(x,y;a).$$

This can be written in short as $f(c) = f(b) * f(a)$.

A Lie groups of point transformations is simply a collection of all such transformations along with this binary composition function that satisfies the four group axioms, written as

(1) For all a, $b \in D$, the domain of the group parameter, there exists an element $c \in D$ such that $f(c) = f(a) * f(b)$.
(2) There exists an element $a_0 \in D$ such that for all $a \in D$, $f(a) * f(a_0) = f(a)$.
(3) For all $a \in D$ there exists an element a^{-1} such that $f(a) * f(a^{-1}) = f(a_0)$.
(4) For all $a, b, c \in D$, $[f(a) * f(b)] * f(c) = f(a) * [f(b) * f(c)]$.

There exist many common transformations that satisfy these four properties and are therefore Lie groups of transformations. These include translations, rotations, scale transformations, Galilean transformations, and Lorentz transformations.

One of the fundamental accomplishments of Lie was to show that for such transformations, all the information concerning the global action of the transformation is contained in the infinitesimal transformation around the identity element a_0. This is a consequence of the continuity properties required for the global transformation functions f_i. Because of this, all the invariance conditions required to construct such transformation groups become linear.

There is a one-to-one correspondence between the global transformation equations using the f_i's and the infinitesimal transformation equations around the identity. The infinitesimal transformation equations are found by expanding a Taylor series around the identity value

$$\tilde{x} = x + (a - a_0)\frac{\partial f_1}{\partial a}\bigg|_{a=a_0} + O[(a - a_0)^2], \quad \tilde{y} = y + (a - a_0)\frac{\partial f_2}{\partial a}\bigg|_{a=a_0} + O[(a - a_0)^2].$$

Special symbols and names are given to these first derivatives of the the transformation equations evaluated at the identity element. They are called the coordinate functions,

$$\frac{\partial f_1}{\partial a}\bigg|_{a=a_0} = \xi(x, y), \quad \frac{\partial f_2}{\partial a}\bigg|_{a=a_0} = \eta(x, y),$$

and they contain complete information about the global transformations since

$$\tilde{x} = \mathrm{e}^{(a-a_0)(\xi\partial_x + \eta\partial_y)}x, \quad \tilde{y} = \mathrm{e}^{(a-a_0)(\xi\partial_x + \eta\partial_y)}y.$$

Here the exponentiation of an operator A formally means

$$\mathrm{e}^{(a-a_0)A}x = \left(1 + (a - a_0)A + \frac{(a - a_0)^2}{2!}(A)(A) + \ldots\right)x.$$

We can also ask how a general function F of x and y changes under the action of the group. We expand F in a Taylor series around the identity as

$$\begin{aligned} F(\tilde{x}, \tilde{y}) =& F(x, y) + (a - a_0)\left[\frac{\partial f_1}{\partial a}\bigg|_{a=a_0}\frac{\partial F}{\partial x} + \frac{\partial f_2}{\partial a}\bigg|_{a=a_0}\frac{\partial F}{\partial y}\right] + O\left[(a - a_0)^2\right], \\ F(\tilde{x}, \tilde{y}) - & F(x, y) = (a - a_0)\left[\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y}\right] \equiv (a - a_0)UF, \end{aligned}$$

where the differential operator

$$U = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

is called the *generator* of the group action. This operator describes how functions change infinitesimally under the action of the group, and it's exponentiation generates the global transformation action,

$$F(\tilde{x}, \tilde{y}) = e^{(a-a_0)U} F(x, y),$$

which is just the Taylor series.

Using this differential operator U we define two types of invariance. A function $F(x, y)$ is said to be an invariant function under the action of the group if $UF = 0$ identically. An equation $F(x, y) = 0$ is an invariant equation if $UF = 0$ whenever $F = 0$. This is written either as $UF|_{F=0} = 0$ or $UF = \lambda(x, y)F$ for some function λ. We now have all the machinery to understand how Lie group theory is used to reduce the space of independent variables for PDE's.

Recall the solution of quasilinear PDE's using the method of characteristics. Given the PDE

$$g_1(\mathbf{x})\frac{\partial F}{\partial x_1} + g_2(\mathbf{x})\frac{\partial F}{\partial x_2} + ... + g_n(\mathbf{x})\frac{\partial F}{\partial x_n} = 0,$$

the general solution is $F = F(c_1, c_2, ..., c_{n-1})$, where the c_i's are the $n-1$ integration constants of the characteristic equations

$$\frac{dx_1}{g_1} = \frac{dx_2}{g_2} = ... = \frac{dx_n}{g_n}.$$

Since the invariance condition $UF = 0$ is this type of PDE, we know that if a function F is invariant under a group with generator $U = \xi\partial x + \eta\partial y$, we can rewrite this function as another function $G(c)$, where c is the integration constant of the characteristic equations

$$\frac{dx}{\xi} = \frac{dy}{\eta},$$

and the dimensionality of the space has been reduced by one.

A simple example is rotation in xy space, where

$$\tilde{x} = x\cos a - y\sin a, \quad \tilde{y} = x\sin a + y\cos a,$$

and the group parameter a is seen to be the angle of rotation. Simple calculation gives $\xi = -y$ and $\eta = x$, so the generator is $U = -y\partial_x + x\partial_y$. (A good excercise is to recover the global equations for $\tilde{x}$ and $\tilde{y}$ from the infinitesimal generator U through $\tilde{x} = e^{aU}x$. $\tilde{y} = e^{aU}y$.) Solving the characteristic equations for this case

$$\frac{dx}{-y} = \frac{dy}{x} \text{ gives } c = x^2 + y^2,$$

and we note any function $F(c)$ is invariant under this group. Therefore, any function $F(x^2 + y^2)$ can be written in terms of this new variable c, which is seen to be the square of the radius. In other words, any function which is rotationally symmetric can be written in terms of the radius only.

The concept of symmetry can be generalized. By definition, if an object is invariant under a transformation it posses a symmetry with respect to that transformation (e.g., reflection and rotation). Differential equations define surfaces (solution surfaces) in space. If this solution surface is invariant under a transformation (it is mapped back into itself), it possesses a symmetry with respect to that transformation. Invariance of the differential equation under the generator U implies a symmetry with respect to the transformation e^{aU}.

With this technique we find we can reduce the number of variables, given invariance under this special linear differential operator U. For a system of PDE's with n independent variables, we can invoke this mechanism $n-1$ times to reduce the system to ODE's, which are then easier to solve. Any solution of the ODE's will provide a particular solution to the original PDE's. Additionally, since the system of equations is invariant, the transformations take solutions into solutions. Often we can use the global transformations to generate a new solution from a given one. Specific examples will be given for the hydrodynamics equations.

3. Hydrodynamics Model

The equations of hydrodynamics can be written

$$\begin{aligned}
&\rho_t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0,\\
&\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla P = 0,\\
&E_t + \mathbf{u}\cdot\nabla E + \frac{1}{\rho}P\nabla\cdot\mathbf{u} + \frac{1}{\rho}\nabla\cdot F = S.
\end{aligned}$$

Here ρ is the material density, $\mathbf{u}$ the 3-dimensional velocity vector whose components can be written (u, v, w), E and P the specific energy and pressure, F is the heat flux, and S is a general energy source term. The equations must be closed with the equation of state, a relationship between energy and pressure, which can be written in terms of a material temperature as

$$E = E(\rho, T), \quad P = P(\rho, T).$$

The heat flux F can be chosen to represent normal material conduction or a nonlinear conduction typical for radiation diffusion, creating Marshak heat fronts. The energy and pressure terms can also be chosen to allow radiation energies and pressures in various forms, such as in a black-body equilibrium form

$$P = \frac{aT^4}{3} \quad \text{and} \quad E = \frac{aT^4}{\rho},$$

a the radiation constant. Radiation conduction can be written

$$F = -\frac{c\lambda}{3}\nabla aT^4.$$

Here c is the speed of light and $\lambda(\rho, T)$ is the radiation mean-free-path.

The Lie group properties of these equations in 1-D including the radiation diffusion terms for an arbitrary material equation of state are listed in Reference 7 along with special cases of reductions to ODE's. For the remainder of this chapter the model will be restricted to a perfect gas EOS with no radiation energy and pressure terms included:

$$\begin{aligned}
&\rho_t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0,\\
&\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla(\Gamma\rho T) = 0,\\
&T_t + \mathbf{u}\cdot\nabla T + (\gamma-1)T\nabla\cdot\mathbf{u} - \frac{\gamma-1}{\Gamma\rho}\nabla\cdot\kappa\nabla T - \frac{\gamma-1}{\Gamma}S = 0.
\end{aligned} \tag{1}$$

Here the material energy and pressure are written in terms of the temperature as

$$P = \Gamma\rho T, \quad E = \frac{\Gamma}{\gamma-1}T,$$

Γ the gas constant and γ the adiabatic exponent. The heat flux F is written in a general diffusion approximation $F = -\kappa(\rho, T)\nabla T$, and can include radiation Marshak behavior.

The Lie groups which leave Equations (1) invariant are generated by the differential operators

$$\begin{aligned}
U_x &= \frac{\partial}{\partial x} \equiv \partial_x,\\
U_y &= \partial_y,\\
U_z &= \partial_z,\\
U_t &= \partial_t,\\
U_{Gx} &= t\partial_x + \partial_u,\\
U_{Gy} &= t\partial_y + \partial_v,\\
U_{Gz} &= t\partial_z + \partial_w,\\
U_{xy} &= -y\partial_x + x\partial_y - v\partial_u + u\partial_v,\\
U_{yz} &= -z\partial_y + y\partial_z - w\partial_v + v\partial_w,\\
U_{zx} &= -x\partial_z + z\partial_x - u\partial_w + w\partial_u,\\
U_{st} &= t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2T\partial_T,\\
U_{ss} &= x\partial_x + y\partial_y + z\partial_z - q\rho\partial_\rho + u\partial_u + v\partial_v + w\partial_w + 2T\partial_T,\\
U_{s\rho} &= \rho\partial_\rho,\\
U_p &= xt\partial_x + yt\partial_y + zt\partial_z + t^2\partial_t - q\rho t\partial_\rho + (x-ut)\partial_u +\\
&\qquad + (y-vt)\partial_v + (z-wt)\partial_w - 2Tt\partial_T,
\end{aligned} \tag{2}$$

with the following conditions for the conductivity κ and energy source S:

$$\begin{aligned}
&\hat{\kappa} = \kappa[a_{s\rho} - a_{st} + (2-q)a_{ss} - qa_p t],\\
&US - S(2a_{ss} - 3a_{st} - 4ta_p) - qa_p\frac{\Gamma T}{\gamma-1}\left(\gamma - \frac{q+2}{q}\right) = 0.
\end{aligned} \tag{3}$$

Here we have written

$$\hat{\kappa} \equiv U\kappa = \eta^\rho \frac{\partial \kappa}{\partial \rho} + \eta^T \frac{\partial \kappa}{\partial T},$$

and expressed the operator U as a linear combination of the 14 separate generators,

$$U = a_x U_x + a_y U_y + \ldots + a_{st} U_{st} + a_{ss} U_{ss} + a_{s\rho} U_{s\rho} + a_p U_p.$$

The first four groups are translations in x, y, z, t. The groups generated by U_{Gi} are Galilean boosts in the x, y, z directions. The next three groups are rotations. Following those are three separate scaling groups in time, space and density. The final group is a projective group in the space-time plane. Note that for the case of no conduction or source S, the conditions (3) become

$$a_p = 0 \quad \text{unless} \quad \gamma = \frac{q+2}{q}.$$

The parameter q takes the value $q = \mathrm{Max}(N, k+1)$, where N is the number of spatial independent variables in the problem and k is a geometry factor equal to 0, 1 or 2 for planar, cylindrical, or spherical geometry, respectively (see Reference 8).

4. One-Dimensional Solutions

In one-dimensional arbitrary coordinates, Equations (1) become

$$\begin{aligned} &\rho_t + u\rho_r + \rho u_r + \frac{k\rho u}{r} = 0, \\ &u_t + uu_r + \frac{1}{\rho}\Gamma T \rho_r + \Gamma T_r = 0, \\ &\frac{\Gamma}{\gamma - 1}(T_t + uT_r) + \Gamma T u_r + \Gamma T \frac{ku}{r} + \frac{1}{\rho}\left(F_r + \frac{kF}{r}\right) = 0. \end{aligned} \tag{4}$$

We let r be the single spatial coordinate for general geometry (planar, cylindrical, spherical), with geometry factor k = 0, 1, or 2. Again, the heat flux can be related to the temperature through a (nonlinear) diffusion approximation $F = -\kappa(\rho, T)T_r$. For the time being we consider the case of no conduction and set $F = 0$.

The groups allowed in one-dimensional coordinates are represented by the generators U_r, U_t, U_{Gr}, U_{st}, U_{ss}, $U_{s\rho}$, and U_p. For the one-dimensional generators we replace x with r and set all terms with y, z, v, w to zero. An additional condition in 1-D occurs for space translation, $ka_r = ka_{Gr} = 0$, which says that translations and Galilean boosts are only allowed in planar geometry ($k = 0$). The condition (3) on γ to keep the projective group in 1-D with no source is $\gamma = (k+3)/(k+1)$.

A. Traditional Similarity Solutions

We note that the "classic" similarity solutions found through dimensional analysis and/or the use of the Buckingham Pi Theorem[9] are exactly the result of the use of scaling groups. The general scaling group allowed for Equations (4) is a linear combination of the three generators U_{st}, U_{ss} and $U_{s\rho}$,

$$U = U_{st} + aU_{ss} + (b + qa)U_\rho = ar\partial_r + t\partial_t + b\rho\partial_\rho + (a-1)u\partial_u + 2(a-1)T\partial_T,$$

a and b free parameters. A group generator can always be arbitrarily scaled, so in the linear combination we can choose one of the multipliers as 1. The global equations for the action of this group are

$$\tilde{r} = e^a r, \quad \tilde{t} = e^1 t, \quad \tilde{\rho} = e^b \rho, \quad \tilde{u} = e^{a-1}u, \quad \tilde{T} = e^{2(a-1)}T.$$

When these tilded variables are substituted into Equations (4), we find the equations remain the same as (4) and are therefore invariant under this scale transformation. Since this U operating on the system (4) is equal to zero on the solution surface of (4), we know that the system can be rewritten solely in terms of the integration constants of the characteristic equations

$$\frac{dr}{ar} = \frac{dt}{t} = \frac{d\rho}{b\rho} = \frac{du}{(a-1)u} = \frac{dT}{2(a-1)T}.$$

Integrating the first relation gives $c_1 = rt^{-a}$, which is identified as the new independent variable s. Integrating the other relations provides the three new dependent variables

$$(c_2 =)H(s) = \frac{\rho}{t^b}, \quad M(s) = \frac{u}{t^{a-1}}, \quad G(s) = \frac{\Gamma T}{t^{2(a-1)}}, \qquad \left(s = \frac{r}{t^a}\right).$$

To transform Equations (4) into these new variables the required derivatives are calculated using the chain rule. For example,

$$\rho = H(s)t^b, \quad \rho_t = H'\frac{\partial s}{\partial t}t^b + bHt^{b-1} = -aH'st^{b-1} + bHt^{b-1}.$$

When these are substituted into Equations (4), they become the ODE's

$$\begin{aligned} &bH + HM' + (M - as)H' + \frac{kHM}{s} = 0 \\ &(a-1)M + (M - as)M' + \frac{H'}{H}G + G' = 0 \\ &2(a-1)G + (M - as)G' + (\gamma - 1)G\left[M' + \frac{kM}{s}\right] = 0 \end{aligned} \tag{5}$$

Many well-known solutions can be constructed from these ODE's, for instance the Sedov and Taylor point explosion,[10] imploding shocks (e.g., Guderley[11]), and a reflecting shock (e.g., Noh[12]). A characteristic of these types of solutions with $M \sim s$ is that the velocity can be written $u(r,t) = u_0 r/t$, which gives material trajectories as $r = r_0|t|^{u_0}$, shown in Figure 1 for $u_0 = 1$. Solutions 0, 1, 2, and 16 in Section 5 are generated using these scaling groups.

B. Exponential solutions

If the translation groups are added to the above scaling groups the solution of the characteristic equations takes two new branches, depending on the choice of parameters. The generator is written

$$U = cU_{st}+aU_{ss}+(b+aq)U_{s\rho}+dU_r+eU_t = (ar+d)\partial_r+(ct+e)\partial_t+b\rho\partial_\rho+(a-c)u\partial_u+2(a-c)T\partial_T.$$

The two new branches are

$$s = \frac{e^{cr/d}}{ct+e}, \quad H(s) = \rho e^{-br/d}, \quad M(s) = u e^{cr/d}, \quad G(s) = \Gamma T e^{2cr/d} \quad \text{for} \quad d, c \neq 0, a = 0,$$

$$s = \frac{e^{at/e}}{ar+d}, \quad H(s) = \rho e^{-bt/e}, \quad M(s) = u e^{-at/e}, \quad G(s) = \Gamma T e^{-2at/e} \quad \text{for} \quad c = 0, e, a \neq 0.$$

The use of scaling groups in conjunction with translation groups gives rise to solutions which are exponential in either time (time translation with space scaling) or space (space translation with time scaling). These solutions have also been found through inspection by many authors.

C. Projective group solutions

The projective group generator

$$U_p = rt\partial_r + t^2\partial_t - q\rho t\partial_\rho + (r-ut)\partial_u - 2Tt\partial_T$$

creates the global transformations

$$\tilde{r} = \frac{r}{1-a_pt}, \; \tilde{t} = \frac{t}{1-a_pt}, \; \tilde{\rho} = \rho(1-a_pt)^{k+1}, \; \tilde{u} = u(1-a_pt)+a_pr, \; \tilde{T} = T(1-a_pt)^2. \quad (6)$$

This is a projective group because these transformation equations take a straight line in $r-t$ space into another straight line, $t = br + c \rightarrow \tilde{t} = b\tilde{r}/(1-a_pc)+c$. This interesting group is allowed only for the special value of the adiabatic exponent $\gamma = (q+2)/q$, which for spherical coordinates is $\gamma = 5/3$.

Keeping the projective group along with density scaling and time translation, we write the generator

$$U = U_p + bU_\rho - a^2U_t = rt\partial_r + (a^2-t^2)\partial_t + (b-qt)\rho\partial_\rho + (r-ut)\partial_u + 2Tt\partial_T.$$

The integration constants of the characteristic equations become the new variables

$$s = \frac{r}{(a^2-t^2)^{1/2}}, \; H(s) = \rho(a^2-t^2)^{q/2}\left(\frac{t+a}{t-a}\right)^{b/2a}, \; M(s) = u(a^2-t^2)^{1/2}-st, \; G(s) = \Gamma T(a^2-t^2).$$

Written in terms of these new variables, the PDE's (4) become the ODE's

$$bH + H'M + HM' + \frac{kHM}{s} = 0$$
$$-a^2 s + M'M + \Gamma\frac{GH'}{H} + \Gamma G' = 0,$$
$$G'M + (\gamma - 1)G\left(M' + \frac{kM}{s}\right) = 0.$$

One can solve these equations with a power law assumption,

$$H(s) = H_0 s^h, \quad M(s) = M_0 s^m, \quad G(s) = G_0 s^g,$$

but the only nontrivial ($M_0 \neq 0$) solution reduces to a special case solution of the ODE's for the scaling groups, so nothing new is obtained. However, making the assumption $M = 0$, the first and third equations reduce to $b = 0$, and the middle equation can be solved in general with only the assumption of isentropic flow (see Reference 13, Appendix C for details) to construct Solution 4. Here the constants R_i and R_o are locations such that at initial time, $\rho(r \leq R_i, 0) = T(r \leq R_i, 0) = 0$, so the solution is for a hollow shell of material initially extending from R_i to R_o. When $R_i = 0$, the solution is no longer hollow and can be written more simply as Solution 3. For the special case of $d = 2\gamma$, this solution becomes Solution 5.

These solutions using the projective group have the interesting property of two singularities on time axis at $t = \pm a$. The material trajectories for Solution 3 are shown in Figure 2. At time $t = -a$, a point explosion occurs and all material expands out from the point $r = 0$ like a "Big Bang". At time $t = a$ all the material collapses again in a "Big Crunch." At the middle point $t = 0$ all material has stopped expanding and the velocity is zero everywhere. This type of solution is also described by Sedov[10] as a pulsating periodic solution.

The projective group can also be used to generate interesting new solutions from known solutions. For instance, Solution 0 is a trivial stationary pressure balance solution. This solution can be transformed to another, nontrivial solution using the global transformation equations (6). We write Solution 0 in tilded variables, $\tilde{\rho}(\tilde{r}, \tilde{t}) = \rho_0 \tilde{r}^b$, $\tilde{u} = 0$, and $\tilde{T}(\tilde{r}, \tilde{t}) = T_0 \tilde{r}^{-b}$, and use the transformation (6) to generate the new solution

$$\tilde{\rho} = \rho(1 - a_p t)^{k+1} = \rho_0 \tilde{r}^b = \rho_0 \left(\frac{r}{1 - a_p t}\right)^b \quad \Rightarrow \quad \rho(r,t) = \rho_0 r^b (1 - a_p t)^{-b-k-1}.$$

Similarly,

$$\tilde{u} = u(1 - a_p t) + a_p r = 0 \quad \Rightarrow \quad u(r,t) = -\frac{a_p r}{1 - a_p t}, \quad \text{and}$$

$$\tilde{T} = T(1 - a_p t)^2 = T_0 \tilde{r}^{-b} = T_0 \left(\frac{r}{1 - a_p t}\right)^{-b} \quad \Rightarrow \quad T(r,t) = T_0 r^{-b} (1 - a_p t)^{b-2}.$$

This solution is identical with Solution 1 after the time translation $\tilde{t} = t - 1/a_p$ is used.

The projective group can often be used to generate new solutions in this fashion. If we have a solution for which $u(r,t) = u_0 r/t$, the material trajectories are found by integrating $\dot{r} = u$ which provides $r(t) = r_0|t|^{u_0}$. Under the action of the projective group with parameter a, these relations become

$$u(r,t) = \frac{r(u_0 + at)}{t(1 + at)}, \quad r(t) = r_0|t|^{u_0}|1 + at|^{1-u_0}.$$

Figures 3 – 5 show the action of the projective group for various values of u_0. It is seen that when $u_0 = 1$ there is no change of the flow. The trivial solution $u_0 = 0$ (Solution 0) goes into a linear imploding/exploding trajectory (Solution 1). Solution 2 has the value $u_0 = 1/2$ for the required γ value for the projective group $\gamma = (k+3)/(k+1)$. This solution is transformed by the projective group into Solution 3 (see Figure 5).

It should be noted that the process of generating a new solution from a given one can only be performed once. A second similar transformation with a different group element gives only a single transformation with a combined composition function value of the two group elements, as is known from the closure property. Secondly, when a particular solution has been obtained as a solution to the ODE's constructed using a certain group, the use of the group again to generate a new solution gives an identity transformation. This is described as follows:

When a subgroup H of the multiparameter invariance group G is used to generate a solution, that solution is called an H-invariant solution. This means that it is invariant under the subgroup H, and any use of the global transformations of H go back into the original solution. To generate a new solution from an H-invariant solution one must use a different transformation from G not in H.

Since a solution can be used to generate new solutions using different groups, it would be nice to identify the minimum collection of subgroups that will generate all possible group invariant solutions. Such a collection is called an optimal system,[1,2,8] and it is constructed by examining the ways in which group invariant solutions transform among themselves through the adjoint operation.

D. Solutions including conduction

Group reduction of the PDE's (4) can be performed including the diffusive conduction term provided the first condition of (3) is met. This condition becomes two relations

$$\rho\kappa_\rho(a_{s\rho} - qa_{ss}) + 2T\kappa_T(a_{ss} - a_{st}) = \kappa[a_{s\rho} - a_{st} + (2-q)a_{ss}], \quad \text{and}$$

$$q\rho\kappa_\rho + 2T\kappa_T = \kappa q,$$

since it is an identity in the variable t. For a power law form of the conductivity, $\kappa = \kappa_0\rho^\alpha T^\beta$, these conditions become (i) a relation between the three scaling parameters, reducing the number of free parameters by one, and (ii) a relationship between α, β, and q if the projective operator is to be used. Generally, one considers the exponents α and β to be given from either theory or data, which means that the projective group usually doesn't exist for conduction. However, one can use the projective group and see what is the resulting relationship between α and β, and decide whether it is useful.

Again, the groups are used to reduce the PDE's (4) to ODE's, for which solutions are sought. It is typical that power law solutions exist, and these generate Solutions 7 – 15. Solution 15 is found with the projective group, and is the conduction companion to Solution 3.

E. Solutions with shocks

Once the groups are used to reduce the PDE's to ODE's, we can consider continuous as well as discontinuous solutions. As an example, consider the ODE's (5) obtained using the scaling groups. For this simple example we will set the parameters $b = 0$ and $a = 1$. Writing these three ODE's in matrix form and solving for the derivatives, we get

$$H' = \frac{kMH(M-s)}{s[(M-s)^2 - \gamma G]},$$
$$M' = \frac{\gamma kMG}{s[(M-s)^2 - \gamma G]},$$
$$G' = \frac{-kMG(\gamma-1)(M-s)}{s[(M-s)^2 - \gamma G]}.$$

Note for $a = 1$ and $b = 0$, $s = r/t$, $H = \rho$, $M = u$, and $G = T$.

A simple solution with a shock can be found by solving these ODE's separately in two regions and connecting the regions with the standard shock jump conditions

$$\rho_1(\dot{R} - u_1) = \rho_2(\dot{R} - u_2) \equiv m_s,$$
$$\Gamma\rho_1 T_1 - \Gamma\rho_2 T_2 = m_s(u_1 - u_2),$$
$$\frac{\Gamma}{\gamma-1}T_1 + \Gamma T_1 + \frac{1}{2}(\dot{R} - u_1)^2 = \frac{\Gamma}{\gamma-1}T_2 + \Gamma T_2 + \frac{1}{2}(\dot{R} - u_2)^2,$$

where R is the shock location and the quantities are evaluated at the shock surface $\pm\epsilon$. The shock location will be stationary in the s coordinate system (required by self-similarity). and the shock location in s space can be called s_1, so $R = s_1 t$, $\dot{R} = s_1$.

We look for a solution describing cold material flowing into the origin with a velocity $u_0(< 0)$, causing a shock to move from the origin outward. There are then two regions: (1), the central region that has experienced the shock, and (2) the outer region with velocity u_0, that the shock is propagating into.

In Region 2, we integrate the ODE's analytically to the shock location, s_1. At $s = \infty$, the density is a constant ρ_0, the velocity is a constant u_0 and the temperature is zero ($T = G = 0$). Since $G = 0$ the last ODE says that G stays zero for all of Region 2. This information in the second ODE says that $M' = 0$, or $M = u_0$ for all Region 2. The first ODE can then be integrated

$$H' = \frac{ku_0 H(u_0 - s)}{s(u_0 - s)^2} \quad \Rightarrow \quad H(s) = H_0\left(\frac{s - u_0}{s}\right)^k.$$

The condition $H(\infty) = \rho_0$ gives $H_0 = \rho_0$. So the solution in Region 2 is

$$H = \rho_0 \left(\frac{s - u_0}{s}\right)^k, \quad M = u_0, \quad G = 0.$$

The jump conditions are now used to connect the solution at the Region 2 side of the shock to just inside Region 1, where the integration will be continued. We find

$$\rho_1 = \rho_2 \frac{s_1 - u_0}{s_1 - u_1}, \quad \Gamma T_1 = (s_1 - u_1)(u_1 - u_0), \quad u_1 = \left\{ \begin{array}{c} 2s_1 - u_0(1-\gamma) \\ -u_0(\gamma + 1) \end{array} \right. .$$

There are two possibilities for u_1 since the jump condition is quadratic. To find out which one is the correct value, we look at the M' equation with the requirement that at the origin the velocity is zero, $M(0) = 0 \quad \Rightarrow \quad M = 0$ for all of Region 1. This fact in the ODE's demands that G and H are constant in Region 1, where they are just $G = T_1$, $H = \rho_1$. Also, since $u_1 = 0$, $s_1 = u_0(1-\gamma)/2$ (the top condition for u_1 is used; the bottom requires $u_0 = 0$) and therefore $\rho_1 = \rho_2(\gamma+1)/(\gamma-1)$. So for Region 1, the solution is

$$H = \rho_1 = \rho_2 \left(\frac{\gamma + 1}{\gamma - 1}\right), \quad M = 0, \quad G = -s_1 u_0.$$

The value of ρ_2 is $H(s_1 + \epsilon)|_{\epsilon=0} = \rho_0[(s_1 - u_0)/s_1]^k = \rho_0[(\gamma+1)/(\gamma-1)]^k$. Putting this all together gives Solution 16. It is interesting that this solution can be projected to produce Solution 17 in the same way that Solution 0 projected produced Solution 1.

More complicated shock solutions are certainly possible. One interesting extension is the addition of conduction, which can be then represented by four first order ODE's (one more being the relationship between the heat flux and the temperature). These equations are generally too difficult to solve analytically, but the ODE's can easily be solved numerically to get similarity solutions including heat conduction. An example of such a solution is in Reference 7.

F. Boundary Conditions

The analytic solutions as given in Section 7 contain no boundary conditions, which must be taken into account for numerical solutions. There are two approaches to this concern. The first and simplest is to consider a finite region initialized with the properties of any of the analytic solutions with no consideration of special boundary conditions. In this approach the evolution at the boundary immediately deviates from the analytic solution, and a rarefaction wave propagates into the material of interest. The solution is valid only in the region which has not felt this rarefaction wave. This approach, while simplest to implement, causes the region of validity to shrink as the problem evolves.

The second approach is to apply the correct boundary conditions at the edge of the problem. This is immediate if the calculation is Eulerian. For a Lagrangian calculation one must calculate the location of the boundary at all times and the appropriate material properties for that location must be imposed. One concern with this approach is that small errors made in these boundary conditions can propagate into the problem and confuse the investigation of internally generated errors. For this reason we generally use the first and simplest approach. More general discussions of the treatment of boundary conditions can be found in References 7 and 13.

5. Two Dimensional Solutions

A. Multiple Reductions using Lie groups

In two dimensions we consider the case of axisymmetry, rotational symmetry around the z axis. This coordinate system can be described in either spherical (r, θ) or cylindrical (R, z) coordinates. There are seven allowed groups in this geometry, which form a 7-parameter group G^7 generated by the operators $(U_z, U_{Gz}, U_{ss}, U_t, U_{st}, U_{s\rho}, U_p)$.

In two dimensions we need to perform two successive group reductions in order to reduce the PDE's to ODE's. We could perform one transformation, reduce the space to 2 independent variables, and then seek the group invariance properties of these reduced equations in order to reduce them a second time to ODE's. Alternatively, it is known that successive reductions are facilitated by looking at the structure of the Lie group via the Lie algebra. Details in the Lie algebra provide information so that invariance properties of the original equations can be inherited by the reduced equations, and the double reduction can be done in one step.

Recall, an algebra is a vector space with a bilinear composition operation $[X, Y]$. A Lie algebra is an algebra for which $[X, Y] = -[Y, X]$. The Lie algebra associated with a Lie group is simply the collection of group generators. The composition operation is the commutator of two generators $[X, Y] = XY - YX$. Lie's principal theorem states that for an n-parameter Lie group with generators U_1, U_2, ..., U_n, the commutator of any two generators is a linear combination of the generators,

$$[U_i, U_j] = \sum_{k=1}^{n} C_{ijk} U_k.$$

The constants C_{ijk} are called the structure constants of the algebra. There is a one-to-one correspondence between the Lie group and the Lie algebra.

A table of commutators can be constructed for any Lie group that allows the identification of subgroups, that is, subsets of generators that are closed under commutation. For this 7-parameter group G^7, the commutator table is shown in Table 1. From this table we can find examples of subgroups. Any collection of generators that commute ($[U_i, U_j] = 0$) form a subgroup, such as $(U_{s\rho}, U_z, U_{Gz})$. This is a commutative or Abelian subgroup, since all commutators are zero. Examples of non-Abelian subgroups are (U_z, U_{Gz}, U_{ss}), (U_z, U_{Gz}, U_p), (U_t, U_{ss}, U_{st}), $(U_t, U_{ss}, U_{st}, U_p)$, etc. Each single generator forms a 1-parameter subgroup.

A special kind of subgroup occurs when the commutator with an element from the subgroup and one from the larger group always goes back into the subgroup. These are called normal subgroups and their corresponding algebras are called ideals.

Definition: Consider a group G and a subset of its elements H along with their associated algebras $\mathcal{G}$ and $\mathcal{H}$. If for any two elements h_1 and h_2 in $\mathcal{H}$, $[h_1, h_2]$ is a linear combination of elements in $\mathcal{H}$, then H is a ***subgroup*** of G and $\mathcal{H}$ is a ***subalgebra*** of $\mathcal{G}$. If for any element h of $\mathcal{H}$ and any element g of $\mathcal{G}$, $[h, g]$ is a linear combination of elements in $\mathcal{H}$, then H is a ***normal subgroup*** of G and $\mathcal{H}$ is an ***ideal*** of $\mathcal{G}$.

All subgroups whose commutators with the larger group are zero are normal subgroups. Examples of normal subgroups are $(U_{st}) \subset_n (U_{st}, U_{ss})$, $(U_{Gz}, U_t) \subset_n (U_{Gz}, U_t, U_z)$, $(U_t) \subset_n (U_t, U_{ss}, U_{st}, U_{s\rho})$, etc.

For any subgroup H of G we can identify the collection of all operators v_i whose commutation with any element in $\mathcal{H}$ goes back into $\mathcal{H}$. This collection, which must contain all of H, is called the ***normalizer*** $\mathrm{Nor}_G(H)$ of the subgroup H of G. It is the largest subgroup of G with the property that H is a normal subgroup of $\mathrm{Nor}_G(H)$. The corresponding algebra $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ is the collection $v_i : [v_i, v_j] \in \mathcal{H} \; \forall \; v_j \in \mathcal{H}, v_i \in \mathcal{G}$.

The theorem pertaining to sequential reductions can now be stated:

Theorem Consider a system of differential equations E invariant under a multiparameter group G with subgroup H. The system E/H obtained by reducing E with the subgroup H will be invariant under the quotient group $Q = \mathrm{Nor}_G(H)/H$.

The quotient group can be formed by simply removing one of the elements of H from $\mathrm{Nor}_G(H)$. A simple proof of this theorem can be found in Ovsiannikov[1982].

To summarize the procedure for multiple reductions, first choose a subgroup H for a reduction. Next, form the normalizer $\mathrm{Nor}_G(H)$ of this collection by finding all the other generators that, when commuted with the collection H, go back into H. Finally, remove any one of the generators of H from the set $\mathrm{Nor}_G(H)$. The resulting collection is the quotient group $Q = \mathrm{Nor}_G(H)/H$. Now use any element from H for the first reduction. One is then guaranteed to be able to do a second reduction using any element of Q.

Therefore, for any two generators U_1 and U_2 which could be linear combinations of single generators, when $[U_1, U_2] = U_1$, use any part of U_1 for the first reduction and then U_2 for the second reduction. It is best to use a part of U_1 that is not included in U_2. More detailed discussion on this process can be found in Reference 8. In that paper is listed the 2-Dimensional optimal system Θ_2, which is a collection of possible ways to do such double reductions. The collection is "optimal" in the sense that all possible double group reduction solutions can be found from the members of this list. Each member provides a path for reducing the 2-D hydro PDE's to ODE's.

B. Reductions to ODE's

Tables 2 and 3 list the new variables for each entry of Θ_2 for the 2-D form of Equations (1) with no conduction or source S. Each entry of Tables 2 and 3 reduces the 2-D PDE's to ODE's. These variables are sometimes easier to write in spherical coordinates (Table 2) and sometimes in cylindrical (Table 3).

As an example, consider the similarity variables found using the reduction path $\mathcal{H}_7$ from Table 2, which uses first the combination $U_1 = U_{st} + \alpha U_{s\rho}$, and next $U_2 = U_{ss} +$

$(\beta+3)U_{s\rho}$, α and β arbitrary constants. To perform a multiple reduction we could first reduce by U_1, write out the new PDE's (which now only have 2 independent variables), and then reduce again with U_2 to ODE's. We know the reduced, intermediate PDE's in two independent variables will be invariant under U_2 since we chose U_1 and U_2 by the above theorem to have that property. We find that it is not necessary to write out the intermediate PDE's; we can go directly from the original PDE's in three independent variables to ODE's in one transformation step as follows.

In spherical coordinates, we find

$$U_1 = t\partial_t + \alpha\rho\partial_\rho - u\partial_u - v\partial_v - 2T\partial_T, \quad \text{and}$$

$$U_2 = r\partial_r + \beta\rho\partial_\rho + u\partial_u + v\partial_v + 2T\partial_T.$$

We first calculate the group invariants of the first generator U_1 by solving the characteristic equations

$$\frac{dr}{0} = \frac{d\theta}{0} = \frac{dt}{t} = \frac{d\rho}{\alpha\rho} = \frac{du}{-u} = \frac{dv}{-v} = \frac{dT}{-2T}$$

for the integration constants $a_1 = r$, $a_2 = \theta$, $a_3 = \rho t^{-\alpha}$, $a_4 = ut$, $a_5 = vt$, $a_6 = Tt^2$. We now write the second generator in terms of these new variables (the a_i's) as

$$U_2 = f_1\partial_{a_1} + f_2\partial_{a_2} + f_3\partial_{a_3} + f_4\partial_{a_4} + f_5\partial_{a_5} + f_6\partial_{a_6},$$

where the f_i's are as of yet unknown functions of the a_i's. The f_i's are found by noticing that $f_i = U_2 a_i$. We calculate

$$f_1 = U_2 a_1 = U_2 r = r = a_1, \quad f_2 = U_2 a_2 = 0, \quad f_3 = U_2 a_3 = U_2(\rho t^{-\alpha}) = \beta\rho t^{-\alpha} = \beta a_3, \text{ etc.}$$

Continuing, we find we can write U_2 completely in terms of the new variables as

$$U_2 = a_1\partial_{a_1} + \beta a_3\partial_{a_3} + a_4\partial_{a_4} + a_5\partial_{a_5} + 2a_6\partial_{a_6}.$$

We now solve the characteristic equations for this operator

$$\frac{da_1}{a_1} = \frac{da_2}{0} = \frac{da_3}{\beta a_3} = \frac{da_4}{a_4} = \frac{da_5}{a_5} = \frac{da_6}{2a_6}$$

for the integration constants $b_1 = a_2 = \theta$, $b_2 = a_3 a_1^{-\beta} = H(\theta)$, $b_3 = a_4/a_1 = M(\theta)$, $b_4 = a_5/a_1 = V(\theta)$, $b_5 = a_6/a_1^2 = G(\theta)$. The integration constants b_i are group invariants of both U_1 and U_2, and are therefore the new variables we are looking for that will reduce the PDE's to ODE's. These new variables H, M, V, and G are the new dependent variables and the new independent variable θ is itself an invariant under both U_1 and U_2. In the same manner as the single reduction in Section 4A, we calculate the derivatives of the old variables in terms of the new variables

$$\rho = H(\theta)r^\beta t^\alpha, \quad \rho_t = \alpha H r^\beta t^{\alpha-1}, \quad \rho_r = \beta H r^{\beta-1} t^\alpha, \quad \rho_\theta = H' r^\beta t^\alpha, \ldots$$

When these are substituted into the PDE's (1) they become the ODE's

$$\alpha + M(\beta+3) + V\frac{H'}{H} + V' + \frac{V}{\tan\theta} = 0,$$
$$M^2 - M + VM' - V^2 + (\beta+2)G = 0,$$
$$-V + 2MV + VV' + G\frac{H'}{H} + G' = 0,$$
$$M(2+3\gamma-3) - 2 + \frac{VG'}{G} + (\gamma-1)\left(V' + \frac{V}{\tan\theta}\right) = 0.$$

Any technique can be attempted to solve the reduced ODE's, which then provides specific solutions for the hydro PDE's. Solutions 18 – 24 are some solutions to the 2-D hydro equations found by making some ansatz in the solution of the associated reduced ODE's.

We find that we can transform Solution 19 with the projective group to get Solution 25. Just as occurred the 1-D solutions, a solution with one pole on the $r-t$ time axis is transformed into one with two poles (see Figure 5). Solution 25 is an explosion/collapse ellipsoidal 2-D solution.

6. 3-D Solutions

The 2-dimensional optimal system Θ_2 was worked out in Reference 8, which yields the minimal reduction paths for the 2-D PDE's into ODE's for which all possible group-invariant solutions may be found. The corresponding 3-D optimal system Θ_3 for the paths for the 3-D PDE's into ODE's has not been worked out. However, guided by the multiple reduction theorem, we can choose a few reduction paths and look for particular solutions. One such path is $U_1 = U_{st} + c_1 U_{ss} + c_2 U_{xy} + c_3 U_{s\rho}$, $U_2 = U_{st} + c_4 U_{ss} + c_5 U_{s\rho}$, $U^3 = U_{st} + c_6 U_{s\rho}$. This gives the similarity variables

$$\rho = H(\theta) t^a r^b e^{c\phi}, \quad u = M(\theta)\frac{r}{t}, \quad v = V(\theta)\frac{r}{t}, \quad w = W(\theta)\frac{r}{t}, \quad \Gamma T = G(\theta)\frac{r^2}{t^2}.$$

When these new variables are substituted into the PDE's (1), they become the ODE's

$$a + M(b+3) + V\frac{H'}{H} + V' + \frac{V}{\tan\theta} + c\frac{W}{\sin\theta} = 0$$
$$-M + M^2 + VM' - V^2 - W^2 + G(b+2) = 0$$
$$-V + 2MV + VV' - \frac{W^2}{\tan\theta} + G\frac{H'}{H} + G' = 0$$
$$-W + 2MW + VW' + \frac{VW}{\tan\theta} + c\frac{G}{\sin\theta} = 0$$
$$-2 + M(2+3\gamma-3) + V\frac{G'}{G} + (\gamma-1)V' + (\gamma-1)\frac{V}{\tan\theta} = 0.$$

A particular solution for these equations can be found through the ansatz $V = 0$, $H = H_0 + H_1(\sin\theta)^d$, $c = 0$, $\gamma = 5/3$, and yields Solution 30. Figure 6 shows material trajectories for Solution 30, which has flow spinning either into or out of the origin (depending on whether time is positive or negative). Figure 7a, b shows the angular profiles for the density, temperature, and ϕ-velocity (w) for this solution with two choices of the free parameters.

An interesting property of this solution is that it can be transformed with the projective group into a new solution, Solution 31, in the same way as Solution 19 gave Solution 25. The spinning behavior and angular profiles are similar to Figures 6 and 7. The radial expansion of these spinning solutions is shown in Figure 5. Figure 5a shows the radial behavior of Solution 30, which either expands or contracts, depending on whether time is positive or negative. Figure 5b shows the radial expansion of Solution 31, which is a point explosion, point collapse spinning 3-D solution.

7. Analytic Solutions

A. 1-D General coordinates, $k = 0,\ 1,\ 2$ for planar, cylindrical or spherical coordinates

0.

$$\begin{aligned} \rho(r,t) &= \rho_0 r^b, \\ u(r,t) &= 0, \\ T(r,t) &= T_0 r^{-b}. \end{aligned}$$

Free parameters: $b,\ \rho_0,\ T_0$

1.

$$\begin{aligned} \rho(r,t) &= \rho_0 r^b t^{-b-k-1}, \\ u(r,t) &= \frac{r}{t}, \\ T(r,t) &= T_0 r^{-b} t^{b-(\gamma-1)(k+1)}. \end{aligned}$$

Free parameters: $b,\ k,\ \rho_0,\ T_0$

2.

$$\rho(r,t) = \rho_0 r^b t^{\frac{-2(b+k+1)}{2+(\gamma-1)(k+1)}},$$
$$u(r,t) = \frac{2}{2+(\gamma-1)(k+1)}\frac{r}{t},$$
$$T(r,t) = \frac{2(\gamma-1)(k+1)}{\Gamma(b+2)[(\gamma-1)(k+1)+2]^2}\frac{r^2}{t^2}.$$

Free parameters: b, k, ρ_0

3.

$$\rho(r,t) = \rho_0 \frac{r^b}{(\tau^2-t^2)^{(k+1+b)/2}},$$
$$u(r,t) = \frac{-rt}{\tau^2-t^2},$$
$$T(r,t) = \frac{\tau^2 r^2}{\Gamma(b+2)(\tau^2-t^2)^2},$$

with

$$\gamma = \frac{k+3}{k+1}.$$

Free parameters: b, k, τ, ρ_0

4.

$$\rho(r,t) = \frac{R_o^{b/\gamma}\tau^{\frac{(k+1)\gamma-1-b}{\gamma-1}} r^{-k-1}}{\left[R_o^{2-b/\gamma}-R_i^{2-b/\gamma}\right]^{\frac{1}{\gamma-1}}}\left(\frac{r}{(\tau^2-t^2)^{1/2}}\right)^{k+1-b/\gamma}\left[\left(\frac{r}{(\tau^2-t^2)^{1/2}}\right)^{2-b/\gamma}-\left(\frac{R_i}{\tau}\right)^{2-b/\gamma}\right]^{\frac{1}{\gamma-1}},$$
$$u(r,t) = \frac{-rt}{\tau^2-t^2},$$
$$T(r,t) = \frac{\tau^2(\gamma-1)r^{-2}}{\Gamma(2\gamma-b)}\left(\frac{r}{(\tau^2-t^2)^{1/2}}\right)^{2+b/\gamma}\left[\left(\frac{r}{(\tau^2-t^2)^{1/2}}\right)^{2-b/\gamma}-\left(\frac{R_i}{\tau}\right)^{2-b/\gamma}\right],$$

with

$$\gamma = \frac{k+3}{k+1}.$$

Free parameters: b, k, τ, R_o, R_i

5.

$$
\begin{aligned}
\rho(r,t) &= \frac{(a^2-t^2)^{(1-k)/2}}{r^2}\left[\rho_0 + \frac{a^2(\gamma-1)}{T_0\gamma}\log\frac{r}{(a^2-t^2)^{1/2}}\right]^{1/(\gamma-1)}, \\
u(r,t) &= \frac{-rt}{a^2-t^2}, \\
T(r,t) &= T_0\frac{r^2}{(a^2-t^2)^2}\left[\rho_0 + \frac{a^2(\gamma-1)}{T_0\gamma}\log\frac{r}{(a^2-t^2)^{1/2}}\right].
\end{aligned}
$$

Free parameters: a, k, ρ_0, T_o

6. With conduction

$$
\begin{aligned}
\rho(r,t) &= \rho_0 r^{\frac{k-1}{\beta-\alpha+4}} t^{-k-1-\frac{k-1}{\beta-\alpha+4}}, \\
u(r,t) &= \frac{r}{t}, \\
T(r,t) &= T_0 r^{\frac{1-k}{\beta-\alpha+4}} t^{(1-\gamma)(k+1)+\frac{k-1}{\beta-\alpha+4}}.
\end{aligned}
$$

Free parameters: α, β, k, ρ_0, T_0

7. With conduction

$$
\begin{aligned}
\rho(r,t) &= \rho_0 r^{-(2\beta+k+7)/\alpha} t^{\frac{-2[\alpha(k+1)-2\beta-k-7]}{\alpha[2+(\gamma-1)(k+1)]}}, \\
u(r,t) &= \frac{2}{2+(\gamma-1)(k+1)}\frac{r}{t}, \\
T(r,t) &= \frac{2\alpha(\gamma-1)(k+1)}{\Gamma[2+(\gamma-1)(k+1)]^2(2\alpha-2\beta-k-7)}\frac{r^2}{t^2}.
\end{aligned}
$$

Free parameters: α, β, k, ρ_0

8. With conduction

$$
\begin{aligned}
\rho(r,t) &= \rho_0 r^{-k}, \\
u(r,t) &= \frac{4c\lambda_0 a}{3}\frac{\gamma-1}{\Gamma\gamma}k\rho_0^{\alpha-1}T_0^{\beta+3}, \\
T(r,t) &= T_0 r^k,
\end{aligned}
$$

with

$$\alpha = \beta + 4 - \frac{1}{k},\ k \neq 0.$$

Free parameters: $\beta,\ k\ , \rho_0,\ T_0$

9. With conduction

$$\begin{aligned}
\rho(r,t) &= \rho_0 r^{(\gamma-1)(k+1)-2} t^{1-k-(\gamma-1)(k+1)}, \\
u(r,t) &= \frac{r}{t}, \\
T(r,t) &= T_0 r^{2-(\gamma-1)(k+1)} t^{-2},
\end{aligned}$$

with

$$\alpha = \beta + 4 + \frac{k-1}{2-(\gamma-1)(k+1)}.$$

Free parameters: $k,\ \rho_0,\ T_0$, either α or β

10. With conduction

$$\begin{aligned}
\rho(r,t) &= \rho_0 r^{\frac{2}{\alpha-\beta-4}} t^{\frac{-2}{\alpha-\beta-4}-k-1}, \\
u(r,t) &= \frac{r}{t}, \\
T(r,t) &= T_0 r^{\frac{-2}{\alpha-\beta-4}} t^{\frac{\alpha(k+1)-k-2}{\beta+3}+\frac{2(\alpha-1)}{(\beta+3)(\alpha-\beta-4)}},
\end{aligned}$$

with

$$T_0 = \left[\frac{3\Gamma}{4c\lambda_0 a(\gamma-1)} \frac{\alpha-1+(\beta+3)(\gamma-1)}{\beta+3} \rho_0^{1-\alpha} \frac{\beta+4-\alpha}{2}\right]^{\frac{1}{\beta+3}}.$$

Free parameters: $\alpha,\ \beta,\ k,\ \rho_0$

11. With conduction

$$\begin{aligned}
\rho(r,t) &= \rho_0 r^{-k-b}, \\
u(r,t) &= r^b \sqrt{\Gamma T_0 \frac{k-b}{b}}, \\
T(r,t) &= T_0 r^{2b},
\end{aligned}$$

with

$$b = \frac{k-1-\alpha k}{2+\alpha-2(\beta+4)}, \text{ and}$$

$$T_0 = \left\{ \frac{b}{\Gamma(k-b)} \left(\frac{4c\lambda_0 a}{3} \frac{\gamma-1}{\Gamma} \right)^2 \frac{\rho_0^{2\alpha-2} 16 b^4}{[2b+(\gamma-1)(k+b)]^2} \right\}^{\frac{-1}{5+2\beta}} .$$

Free parameters: $\alpha,\ \beta,\ k,\ \rho_0$

12. With conduction

$$\begin{aligned} \rho(r,t) &= \rho_0 r^{\frac{2}{\alpha-\beta-4}} t^{-k-1-\frac{2}{\alpha-\beta-4}}, \\ u(r,t) &= \frac{r}{t}, \\ T(r,t) &= T_0 r^{\frac{-2}{\alpha-\beta-4}} t^{-2}, \end{aligned}$$

with

$$\frac{2}{\alpha-\beta-4} = \frac{k+4-\alpha(k+1)-2(\beta+4)}{\alpha-1}, \text{ and}$$

$$T_0 = \left[\frac{3}{4c\lambda_0 a} \frac{\Gamma}{2(k+1)(\gamma-1)} \rho_0^{1-\alpha} \left\{ 2 + [2-(\gamma-1)(k+1)]\,(\alpha-\beta-4) \right\} \right]^{\frac{1}{\beta+3}} .$$

Free parameters: $k,\ \rho_0$, either α or β

13. With conduction

$$\begin{aligned} \rho(r,t) &= \rho_0 r^{-k-b}, \\ u(r,t) &= u_0 r^b, \\ T(r,t) &= \frac{u_0^2 b}{\Gamma(k-b)} r^{2b}, \end{aligned}$$

with

$$\alpha = 1 - \frac{1}{k},\ \beta = \frac{1}{2}\alpha - 3,\ k \neq 0, \text{ and}$$

$$\rho_0 = \left[\frac{16 c\lambda_0 a}{3} (\gamma-1) \right]^k \frac{b^{(5k-1)/2}}{u_0 (k-b)^{(k-1)/2} \Gamma^{(3k-1)/2} [2b+(\gamma-1)(k+b)]^k} .$$

Free parameters: $b,\ k,\ u_0$

14. With conduction

$$
\begin{aligned}
\rho(r,t) &= \rho_0 r^{\frac{2\beta-4}{1-\alpha}} t^{\frac{2\beta+5}{\alpha-1}}, \\
u(r,t) &= u_0 \frac{r}{t}, \\
T(r,t) &= T_0 \frac{r^2}{t^2},
\end{aligned}
$$

with

$$
\begin{aligned}
u_0 &= \frac{2\beta+5}{2\beta-4+(1-\alpha)(k+1)}, \\
T_0 &= \frac{(\alpha-1)(2\beta+5)[9-(1-\alpha)(k+1)]}{\Gamma[2\beta-4+(1-\alpha)(k+1)]^2[2\beta-4+2(1-\alpha)]}, \\
\rho_0 &= \left[\frac{-2+u_0[2+(\gamma-1)(k+1)]}{2\left[\frac{\alpha}{1-\alpha}(2\beta-4)+2\beta+k+7\right]} \frac{3}{4c\lambda_0 a} \frac{\Gamma}{\gamma-1} T_0^{-\beta-3} \right]^{\frac{1}{\alpha-1}}.
\end{aligned}
$$

Free parameters: $\alpha,\ \beta,\ k$

15. With conduction

$$
\begin{aligned}
\rho(r,t) &= \rho_0 r^{-\frac{1}{\alpha}(2\beta+k+7)} (\tau^2-t^2)^{-\frac{1}{2}(k+1)+\frac{1}{2\alpha}(2\beta+k+7)}, \\
u(r,t) &= \frac{-rt}{\tau^2-t^2}, \\
T(r,t) &= \frac{\alpha\tau^2}{\Gamma(2\alpha-2\beta-k-7)} \frac{r^2}{(\tau^2-t^2)^2},
\end{aligned}
$$

with

$$
\gamma = \frac{k+3}{k+1}.
$$

Free parameters: $\alpha,\ \beta,\ k,\ \tau,\ \rho_0$

16. Shock, no conduction

Region 1:

$$
\begin{aligned}
\rho(r,t) &= \rho_0 \left(\frac{\gamma+1}{\gamma-1}\right)^{k+1}, \\
u(r,t) &= 0, \\
T(r,t) &= \frac{u_0^2(\gamma-1)}{2\Gamma},
\end{aligned}
$$

Region 2:

$$\rho(r,t) = \rho_0 \left(\frac{r - u_0 t}{r}\right)^k,$$
$$u(r,t) = u_0 (< 0),$$
$$T(r,t) = 0.$$

The shock location is

$$R = -\frac{1}{2}(\gamma - 1) u_0 t.$$

Free parameters: $k,\ u_0,\ \rho_0$

17. Shock, no conduction

Region 1:

$$\rho(r,t) = \rho_0 \left(\frac{\gamma + 1}{\gamma - 1}\right)^{k+1} (1 - at)^{-k-1},$$
$$u(r,t) = -\frac{ar}{1 - at},$$
$$T(r,t) = \frac{u_0^2(\gamma - 1)}{2\Gamma}(1 - at)^{-2},$$

Region 2:

$$\rho(r,t) = \rho_0 (1 - at)^{-k-1} \left(\frac{r - u_0 t}{r}\right)^k,$$
$$u(r,t) = \frac{u_0 - ar}{1 - at},$$
$$T(r,t) = 0.$$

The shock location is

$$R = \frac{u_0(\gamma - 1)}{4a} \frac{t(1 - 2at)}{1 - at}$$

$$\text{with } \gamma = \frac{k+3}{k+1}.$$

Free parameters: $a,\ k,\ \rho_0,\ u_0$

B. 2-D axisymmetric flow, (r, θ) or (R, z):

Solutions 18 – 21 come from $\mathcal{H}_7$ with the ansatz $M = M_0(a + b\cos^2\theta),\ V = V_0 \sin\theta\cos\theta$.

18. $\beta \neq -2,\ bM_0 + V_0 = 0,\ aM_0 = 0$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 t^{-2(\beta+1)/(\gamma+1)} \left(r\cos\theta\right)^{\beta} \\
u(r,\theta,t) &= \frac{2}{(\gamma+1)}\frac{r}{t}\cos^2\theta \\
v(r,\theta,t) &= -\frac{2}{(\gamma+1)}\frac{r}{t}\sin\theta\cos\theta \\
T(r,\theta,t) &= \frac{2(\gamma-1)}{\Gamma(\gamma+1)^2(\beta+2)}\left(\frac{r}{t}\right)^2\cos^2\theta
\end{aligned}$$

Free parameters: $\beta,\ \rho_0$

19. $\beta \neq -2,\ bM_0 + V_0 = 0, aM_0 = 1$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 t^{-\beta-3+3(\beta+1)(\gamma-1)/(\gamma+1)} \left(r\cos\theta\right)^{\beta} \\
u(r,\theta,t) &= \frac{r}{t}\left(1 - 3\frac{\gamma-1}{\gamma+1}\cos^2\theta\right) \\
v(r,\theta,t) &= 3\frac{\gamma-1}{\gamma+1}\frac{r}{t}\sin\theta\cos\theta \\
T(r,\theta,t) &= \frac{6(\gamma-1)(2-\gamma)}{\Gamma(\gamma+1)^2(\beta+2)}\left(\frac{r}{t}\right)^2\cos^2\theta
\end{aligned}$$

Free parameters: β, ρ_0

20. $\beta = -2,\ aM_0 = 0$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 t^{2-\gamma} r^{-2} \left(\sin\theta\right)^{1-\gamma}\left(\cos\theta\right)^{\gamma-3} \\
u(r,\theta,t) &= \frac{r}{t}\cos^2\theta \\
v(r,\theta,t) &= -\frac{r}{t}\sin\theta\cos\theta \\
T(r,\theta,t) &= T_0\left(\frac{r}{t}\right)^2\left(\sin\theta\right)^{\gamma-1}\left(\cos\theta\right)^{3-\gamma}
\end{aligned}$$

Free parameters: $\rho_0,\ T_0$

21. $\beta = -2,\ aM_0 = 1$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 t^{2(1-\gamma)} r^{-2} \left(\sin\theta\right)^{2\gamma-4}\left(\cos\theta\right)^{2-2\gamma} \\
u(r,\theta,t) &= \frac{r}{t}(1-\cos^2\theta) \\
v(r,\theta,t) &= \frac{r}{t}\sin\theta\cos\theta \\
T(r,\theta,t) &= T_0\left(\frac{r}{t}\right)^2\left(\sin\theta\right)^{4-2\gamma}\left(\cos\theta\right)^{2\gamma-2}
\end{aligned}$$

Free parameters: ρ_0, T_0

22. $\mathcal{H}_{12}$ with the ansatz $A = 0$:

$$\begin{aligned}
\rho(R,z,t) &= \rho_0 \exp\left(\alpha t + \beta z - a\beta t^2\right)\\
u^R(R,z,t) &= 0,\\
u^z(R,z,t) &= -\frac{\alpha}{\beta} + 2at\\
T(R,z,t) &= -\frac{2a}{\Gamma\beta}
\end{aligned}$$

Free parameters: ρ_0, α, β, a

23. $\mathcal{H}_{14}$ with the ansatz $M = 0$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 r^{2[-2a+\gamma(a+1)]/(2-\gamma)} \left(\sin\theta\right)^{-2/(\gamma+1)}\\
u(r,\theta,t) &= 0\\
v(r,\theta,t) &= v_0 r^a \left(\sin\theta\right)^{(1-\gamma)/(1+\gamma)}\\
T(r,\theta,t) &= v_0^2 \frac{1-\gamma}{2\Gamma\gamma} r^{2a} \left(\sin\theta\right)^{(2-2\gamma)/(\gamma+1)}
\end{aligned}$$

Free parameters: ρ_0, a, v_0

24. $\mathcal{H}_{14}$ with the ansatz $M = M_0\cos\theta$, $V = V_0\sin\theta$:

$$\begin{aligned}
\rho(r,\theta,t) &= \rho_0 r^{-2\gamma/(2\gamma-1)} \left(\sin\theta\right)^{(2-2\gamma)/(2\gamma-1)}\\
u(r,\theta,t) &= u_0\cos\theta\\
v(r,\theta,t) &= u_0\frac{1-\gamma}{\gamma}\sin\theta\\
T(r,\theta,t) &= u_0^2\frac{(\gamma-1)(2\gamma-1)}{2\Gamma\gamma^3}\sin^2\theta
\end{aligned}$$

Free parameters: ρ_0, v_0, u_0

25. (Solution 19, projected)

$$\rho(r,\theta,t) = \rho_0(r\cos\theta)^\beta (t-\tau)^{-\beta-3+3(\beta+1)(\gamma-1)/(\gamma+1)}[1+s(t-\tau)]^{-3(\beta+1)(\gamma-1)/(\gamma+1)}$$

$$u(r,\theta,t) = \frac{r}{s\left\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\right\}}\left(1-3\frac{\gamma-1}{\gamma+1}\cos^2\theta\right) + \frac{sr}{1+s(t-\tau)}$$

$$v(r,\theta,t) = \frac{3(\gamma-1)r\sin\theta\cos\theta}{s(\gamma+1)\left\{[t-\tau+1/(2s)]^2 - \frac{1}{4s^2}\right\}}$$

$$T(r,\theta,t) = \frac{6(\gamma-1)(2-\gamma)}{\Gamma(\gamma+1)^2(\beta+2)}\cos^2\theta\left[\frac{r}{s\left\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\right\}}\right]^2,$$

Free parameters: ρ_0, β, s. This solution reduces to Solution 19 when $\gamma \neq 5/3$.

C. **2-D cylindrical solutions** (R, ϕ):

Here the groups $\langle U_{ss} + c_1 U_{st} + c_2 U_{xy} + c_3 U_{s\rho},\ U_{xy} + c_4 U_{ss} + c_5 U_{s\rho}\rangle$ were used to generate the similarity variables

$$\lambda = R^a t^b \mathrm{e}^{-c\phi},\ \rho = H(\lambda)\left(\frac{R}{t}\right)^d t^e,\ u^R = M(\lambda)\frac{R}{t},$$
$$u^\phi = V(\lambda)\frac{R}{t},\ T = G(\lambda)\left(\frac{R}{t}\right)^2.$$

26. $\gamma = 2,\ e = 0$:

$$\rho(R,\phi,t) = \rho_0\left(\frac{R}{t}\right)^2$$
$$u^R(R,\phi,t) = \frac{R}{2t}$$
$$u^\phi(R,\phi,t) = v_0\frac{R}{t}$$
$$T(R,\phi,t) = \frac{4v_0^2+1}{16\Gamma}\left(\frac{R}{t}\right)^2$$

Free parameters: ρ_0, v_0

27. $\gamma = 2,\ e = 0$:

$$\rho(R,\phi,t) = \rho_0 R^{-2}\left(\frac{R}{t}\right)^a$$
$$u^R(R,\phi,t) = \frac{R}{t}$$
$$u^\phi(R,\phi,t) = v_0 R^{-1}\left(\frac{R}{t}\right)^b$$
$$T(R,\phi,t) = \frac{v_0^2}{\Gamma(a+2b-4)}R^{-2}\left(\frac{R}{t}\right)^{2b}$$

Free parameters: ρ_0, v_0, a, b

28.

$$\rho(R,\phi,t) = \rho_0\left(\frac{R}{t}\right)^{d-bf} R^{f(a+b)}\mathrm{e}^{-c\phi}$$
$$u^R(R,\phi,t) = \frac{R}{\gamma t}$$
$$u^\phi(R,\phi,t) = \frac{-d+(2+d+af)/\gamma}{c}\frac{R}{t}$$
$$T(R,\phi,t) = \frac{(2-\gamma)(1-\gamma)R^2}{\Gamma\gamma^2[(\gamma-1)(af+2)+fb\gamma-d]t^2}$$

with $c^2 = \dfrac{\gamma}{1-\gamma}[f(b+a/\gamma)+(d+2)/\gamma)-d][(\gamma-1)(af+2)+fb\gamma-d]$

Free parameters: $\rho_0,\ a,\ b,\ d,\ f$

29. $V = 0$:

$$\rho(R,\phi,t) = \rho_0 \left(\frac{R}{t}\right)^a t^{-2} e^{c\phi}$$
$$u^R(R,\phi,t) = \frac{R}{t}$$
$$u^\phi(R,\phi,t) = 0$$
$$T(R,\phi,t) = T_0 \left(\frac{R}{t}\right)^{-a} t^{2(1-\gamma)} e^{-c\phi}$$

Free parameters: $\rho_0,\ a,\ T_0$

D. 3-D solution, spherical coordinates (r,θ,ϕ):
30. $V = 0$, $H = H_0 + H_1(\sin\theta)^a$, $c = 0$, $\gamma = 5/3$:

$$\rho(r,\theta,\phi,t) = t^{-(b+3)/2} r^b (\rho_0 + \rho_1 \sin^a\theta)$$
$$u(r,\theta,\phi,t) = \frac{r}{2t}$$
$$v(r,\theta,\phi,t) = 0$$
$$w(r,\theta,\phi,t) = \pm\frac{r}{4t}\left[\frac{4\Gamma T_0(b+2)(\sin\theta)^{b+2}}{\rho_0 + \rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2-a)} + \frac{a}{(b+2-a)}\right]^{1/2}$$
$$T(r,\theta,\phi,t) = \left(\frac{r}{4\Gamma t}\right)^2 \left[\frac{4\Gamma T_0(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2)(b+2-a)} + \frac{1}{(b+2-a)}\right]$$

Free parameters: $\rho_0,\ \rho_1,\ T_0,\ a,\ b$

31. (Solution 30, projected; $\gamma = 5/3$)

$$\rho(r,\theta,\phi,t) = [(t-\tau)(1+s(t-\tau))]^{-(b+3)/2} r^b(\rho_0 + \rho_1\sin^a\theta)$$
$$u(r,\theta,\phi,t) = \frac{r}{2s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}} + \frac{sr}{1+s(t-\tau)}$$
$$v(r,\theta,\phi,t) = 0$$
$$w(r,\theta,\phi,t) = \pm\frac{1}{4}\left[\frac{4\Gamma T_0(b+2)(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2-a)} + \frac{a}{b+2-a}\right]^{1/2} \times \frac{r}{s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}}$$
$$T(r,\theta,\phi,t) = \frac{1}{4\Gamma}\left[\frac{4\Gamma T_0(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2)(b+2-a)} + \frac{1}{(b+2-a)}\right] \times \left[\frac{r}{s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}}\right]^2.$$

Free parameters: $\rho_0,\ \rho_1,\ T_0,\ a,\ b,\ s$

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	$U_{s\rho}$	U_z	U_{Gz}	U_{ss}	U_t	U_{st}	U_p
$U_{s\rho}$	0	0	0	0	0	0	0
U_z	0	0	0	U_z	0	0	U_{Gz}
U_{Gz}	0	0	0	U_{Gz}	$-U_z$	$-U_{Gz}$	0
U_{ss}	0	$-U_z$	$-U_{Gz}$	0	0	0	0
U_t	0	0	U_z	0	0	U_t	$U_{ss}+2U_{st}$
U_{st}	0	0	U_{Gz}	0	$-U_t$	0	U_p
U_p	0	$-U_{Gz}$	0	0	$-U_{ss}-2U_{st}$	$-U_p$	0

Table 1. Table of commutators for the subgroup allowed in 2-D axisymmetric geometry

Group	U_1	U_2	new ind. var.	ρ	ΓT	u	v
$\mathcal{H}_{1+}$	$a^2U_t + U_p + \alpha U_{s\rho}$	$U_{ss} + \beta U_{s\rho}$	θ	$Hr^{\beta-3}(a^2+t^2)^{-\beta/2}\exp\left(\frac{\alpha}{a}\tan^{-1}\frac{t}{a}\right)$	$G\frac{r^2}{(a^2+t^2)^2}$	$\frac{U+t}{a^2+t^2}r$	$V\frac{r}{a^2+t^2}$
$\mathcal{H}_{1-}$	$-a^2U_t + U_p + \alpha U_{s\rho}$	$U_{ss} + \beta U_{s\rho}$	θ	$Hr^{\beta-3}(a^2-t^2)^{-\beta/2}\left(\frac{a-t}{a+t}\right)^{\alpha/2a}$	$G\frac{r^2}{(a^2-t^2)^2}$	$\frac{U-t}{a^2-t^2}r$	$V\frac{r}{a^2-t^2}$
$\mathcal{H}_2$	U_p	$U_{ss} + (1-\frac{1}{a})U_{st} + \frac{\beta}{a}U_{s\rho}$	θ	$Hr^{-3}\left(\frac{r}{t}\right)^{\beta}$	$\frac{G}{t^2}\left(\frac{r}{t}\right)^{2a}$	$\frac{U}{t}\left(\frac{r}{t}\right)^{a} + \frac{r}{t}$	$\frac{V}{t}\left(\frac{r}{t}\right)^{a}$
$\mathcal{H}_3$	$U_p + \alpha U_{s\rho}$	$U_{ss} + \beta U_{s\rho}$	θ	$Hr^{\beta-3}t^{-\beta}\mathrm{e}^{-\alpha/t}$	Gr^2/t^4	$\frac{U+t}{t^2}r$	Vr/t^2
$\mathcal{H}_5$	U_p	$U_{st} + \beta U_{s\rho}$	θ	$Ht^{-3}\left(\frac{r}{t}\right)^{-\beta}$	G/t^2	$\frac{U+r}{t}$	V/t
$\mathcal{H}_7$	$U_{st} + \alpha U_{s\rho}$	$U_{ss} + (\beta+3)U_{s\rho}$	θ	$Ht^{\alpha}r^{\beta}$	Gr^2/t^2	Ur/t	Vr/t
$\mathcal{H}_{13}$	$U_t + \alpha U_{s\rho}$	$U_{ss} + (\beta+3)U_{s\rho}$	θ	$Hr^{\beta}\mathrm{e}^{\alpha t}$	Gr^2	Ur	Vr
$\mathcal{H}_{14}$	U_t	$U_{ss} + (a+1)U_{st} + (\beta+3)U_{s\rho}$	θ	Hr^{β}	Gr^{2a}	Ur^{a}	Vr^{a}

Table 2. Members of the 2-dimensional optimal system Θ_2 that are easier to write in spherical coordinates. Each entry in Tables 2 and 3 show the generators U_1, U_2 used for the double reduction and the resulting new variables in which the 2-D hydro equations become ordinary differential equations.

Group	U_1	U_2	newind. var.	ρ	ΓT	u_R	u_z
$\mathcal{H}_4$	U_p	$U_{Gz} + a(U_{ss} + U_{st}) + \beta U_{s\rho}$	R/t	$Ht^{-3}e^{\beta z/t}$	$e^{2az/t}G/t^2$	$(Ae^{az/t} + R)/t$	$(Be^{az/t} + z)/t$
$\mathcal{H}_6$	$U_p + \alpha U_{s\rho}$	$U_{Gz} + \beta U_{s\rho}$	R/t	$Ht^{-3}e^{(\beta z - \alpha)/t}$	G/t^2	$(A + R)/t$	$(B + z)/t$
$\mathcal{H}_8$	$U_{ss} + U_{st} + (\alpha + 3)U_{s\rho}$	$U_{Gz} + \beta U_{s\rho}$	R/t	$Ht^{\alpha}e^{\beta z/t}$	G	A	$B + z/t$
$\mathcal{H}_9$	$U_{st} + \alpha U_{s\rho}$	$U_z + \beta U_{s\rho}$	R	$Ht^{\alpha}e^{\beta z}$	G/t^2	A/t	B/t
$\mathcal{H}_{10}$	$aU_{Gz} + U_{ss} + bU_t + (\alpha + 3 - \beta b)U_{s\rho}$	$aU_z - U_t + \beta U_{s\rho}$	$R/(z + at + b)$	$Hx^{\alpha}e^{-\beta t}$	Gx^2	Ax	$Bx - a$
$\mathcal{H}_{11}$	$2aU_{Gz} + U_t$	$2U_{ss} + U_{st} + 2(\beta + 3)U_{s\rho}$	$R/(z - at^2)$	Hx^{β}	Gx	$Ax^{1/2}$	$Bx^{1/2} + 2at$
$\mathcal{H}_{12}$	$2aU_{Gz} + U_t + \alpha U_{s\rho}$	$U_z + \beta U_{s\rho}$	R	$H\exp\left(\alpha t + \beta z - a\beta t^2\right)$	G	A	$B + 2at$
$\mathcal{H}_{16}$	U_t	$U_z + aU_{st} + \beta U_{s\rho}$	R	$He^{\beta z}$	Ge^{-2az}	Ae^{-az}	Be^{-az}
$\mathcal{H}_{18}$	U_{Gz}	$aU_{ss} + U_p + (\beta + 3a)U_{s\rho}$	$\frac{R}{t}e^{a/t}$	$Ht^{-3}e^{\beta/t}$	$e^{-2a/t}G/t^2$	$\left(Ae^{-a/t} + R\right)/t$	$\left(Be^{-a/t} + z\right)/t$
$\mathcal{H}_{19}$	$U_{Gz} + \alpha U_{s\rho}$	$aU_z + U_p + \beta U_{s\rho}$	R/t	$Ht^{-3}\exp\left(\alpha z/t - 2\alpha a/t^2 + \beta/t\right)$	G/t^2	$(A + R)/t$	$(B + z)/t + a/t^2$
$\mathcal{H}_{20}$	U_{Gz}	$aU_z + U_{st} + \beta U_{s\rho}$	R	Ht^{β}	G/t^2	A/t	$(B + z - a\ln t)/t$
$\mathcal{H}_{22}$	U_{Gz}	$aU_{ss} + U_{st} + (\beta + 3a)U_{s\rho}$	R/t^a	Ht^{β}	$Gt^{2(a-1)}$	At^{a-1}	$(Bt^a + z)/t$
$\mathcal{H}_{23}$	U_z	$aU_{Gz} + U_{ss} + U_{st} + (\beta + 3)U_{s\rho}$	R/t	Ht^{β}	G	A	$B + a\ln t$
$\mathcal{H}_{24}$	U_z	$aU_{ss} + U_{st} + (\beta + 3a)U_{s\rho}$	R/t^a	Ht^{β}	$Gt^{2(a-1)}$	At^{a-1}	Bt^{a-1}
$\mathcal{H}_{25}$	U_z	$aU_{ss} + U_t + (\beta + 3a)U_{s\rho}$	Re^{-at}	$He^{\beta t}$	Ge^{2at}	Ae^{at}	Be^{at}

Table 3. Members of Θ_2 written in cylindrical coordinates (R, z). Each entry reduces the 2-D partial differential equations to ordinary differential equations.

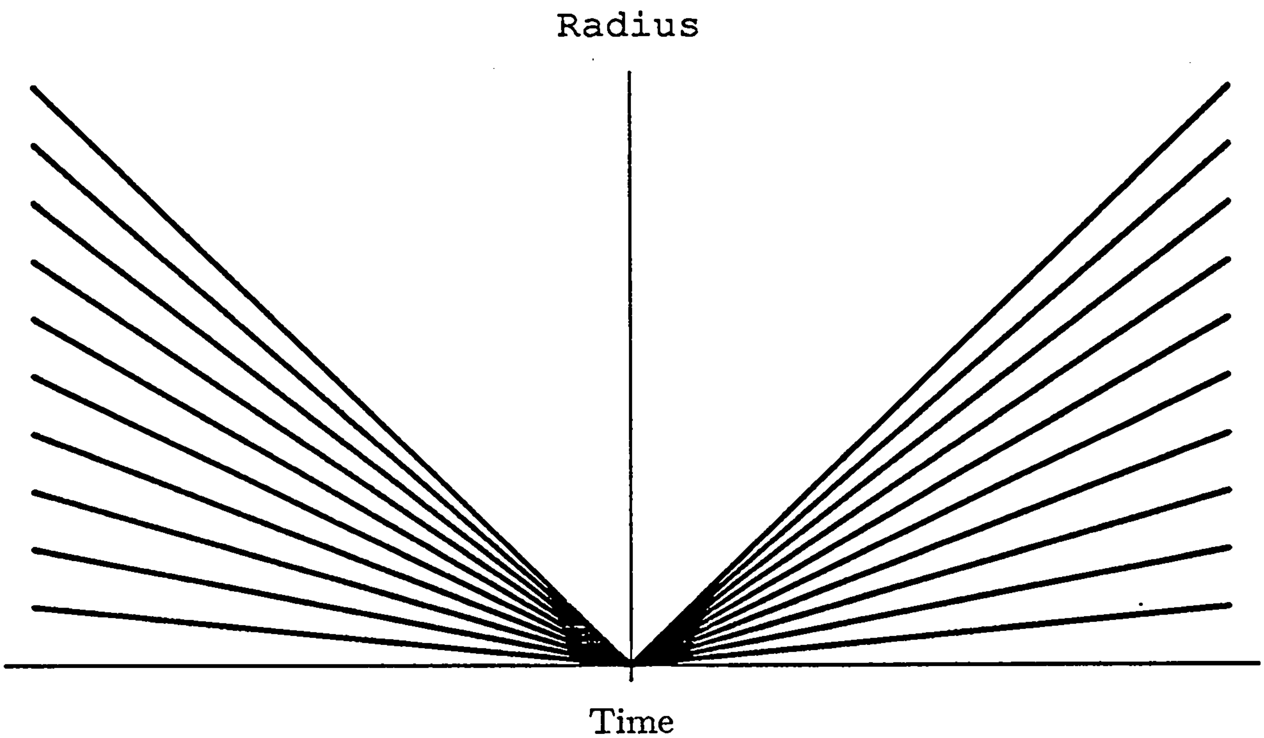


Figure 1. Material trajectories for a solution with $u(r,t) = u_0 r/t$.

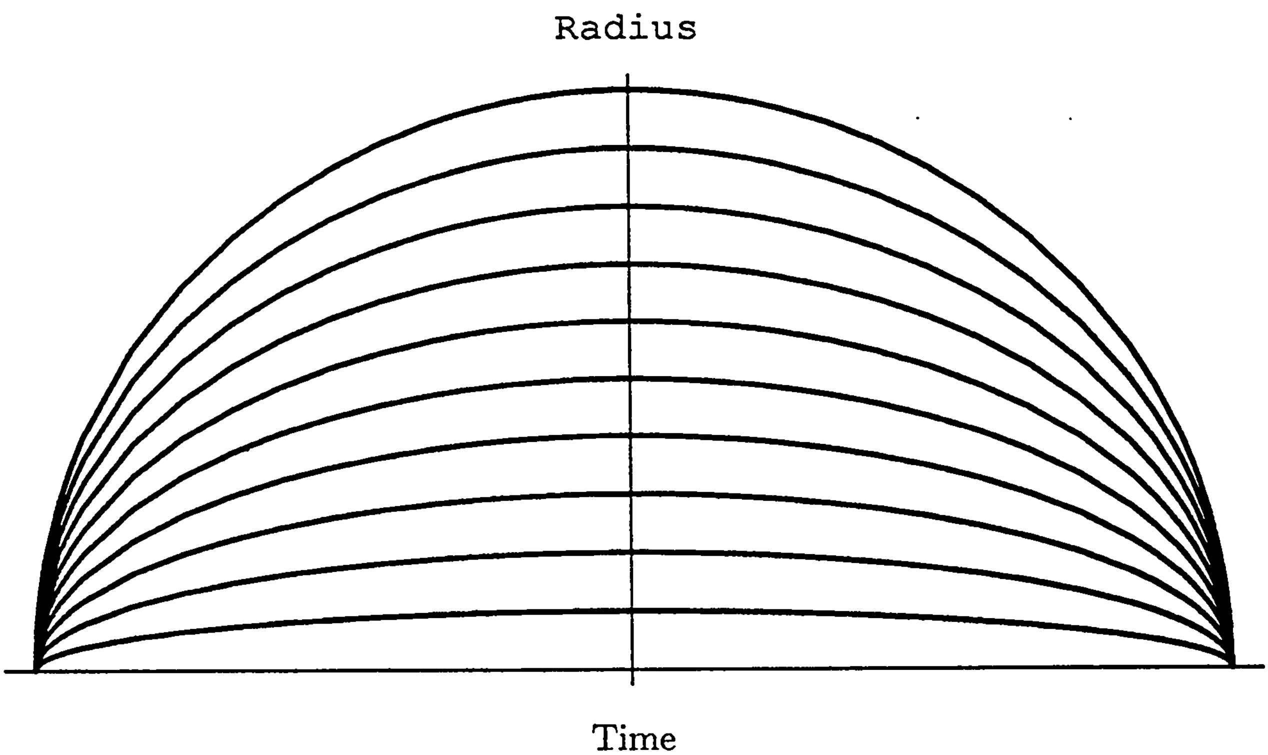
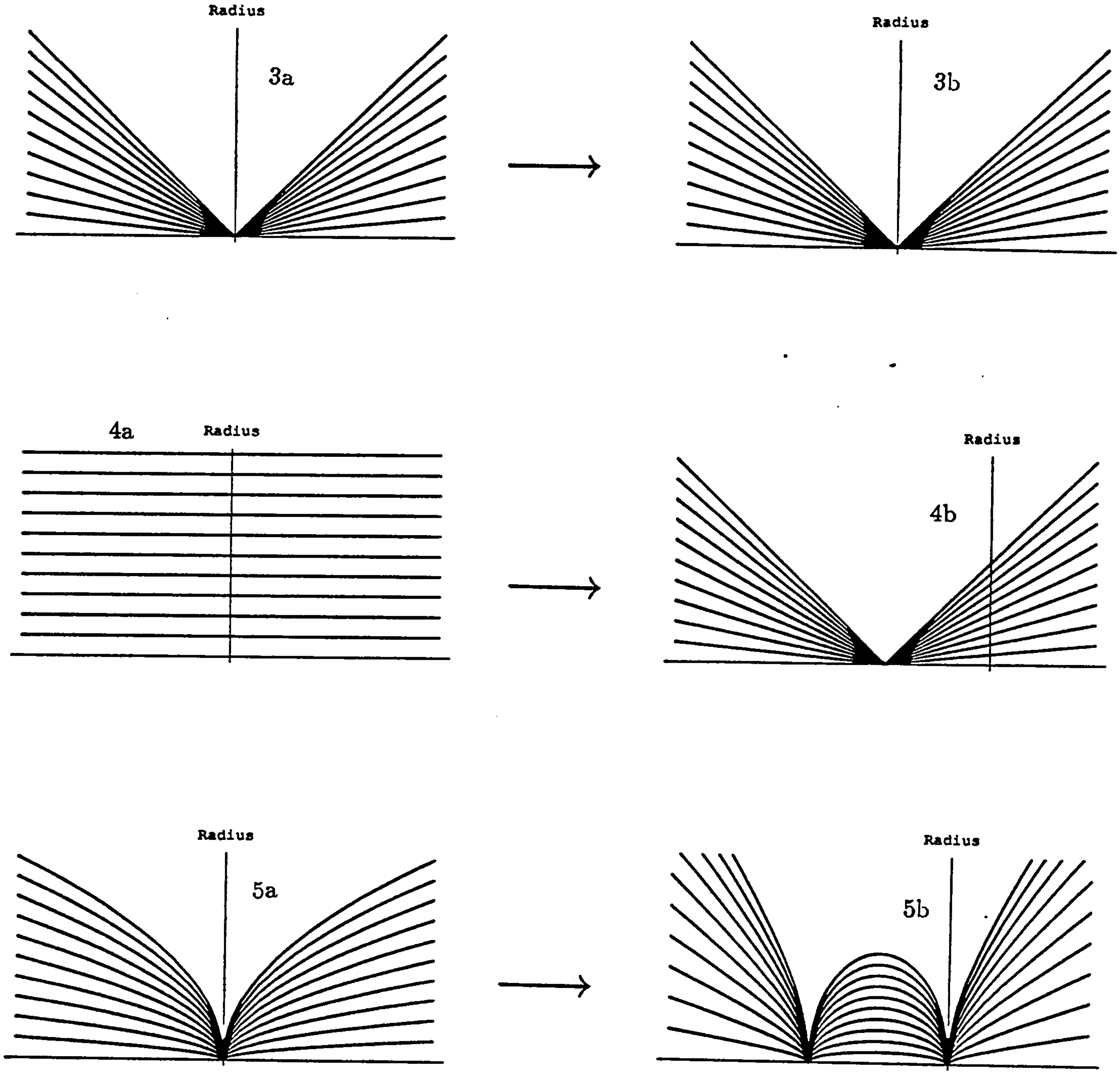


Figure 2. Material trajectories for Solutions 3.



Figures 3 – 5. Figures 3a –5a show material trajectories for solutions with $u(r,t) = u_0 r/t$, with values u_0 =1, 0, and 1/2 respectively. Figures 3b – 5b show the effect of the projective group action on these solutions.

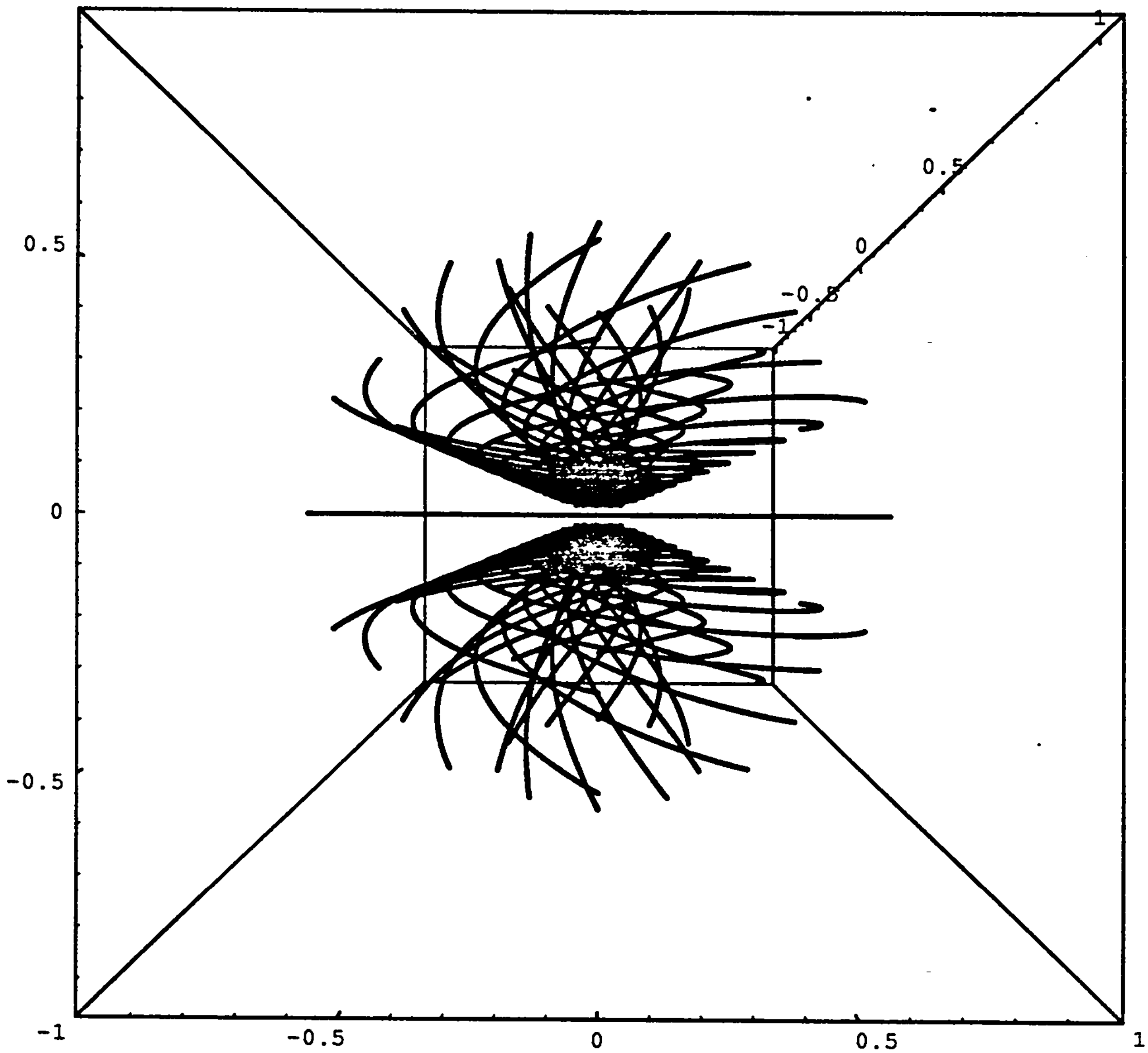


Figure 6. Material trajectories for Solution 30 with $a = 1$ and $b = 2$.

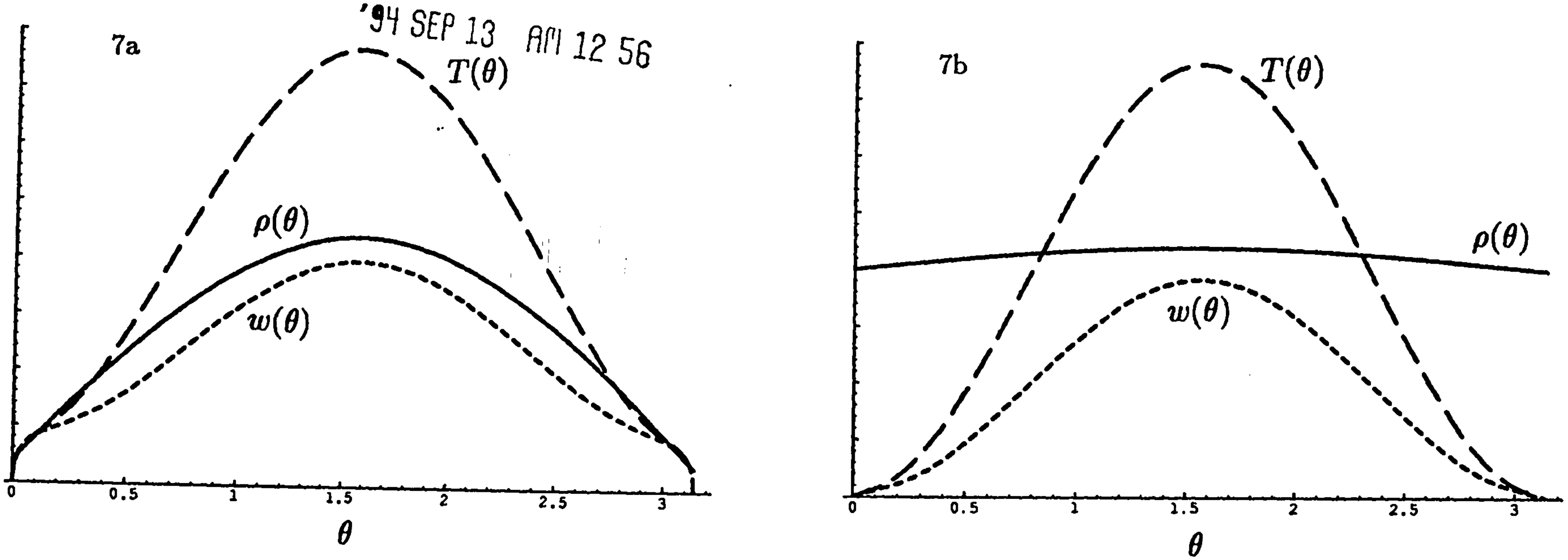


Figure 7. Angular profiles for Solutions 30 and 31 with $a = 1$, $b = 2$. 7a has $\rho_0 = 1$, $\rho_1 = .1$ and 7b uses $\rho_0 = .1$, $\rho_1 = 1$.