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MASTER

AN INSTABILITY LOCALIZED AT THE INNER
SURFACE OF AN IMPLODING SPHERICAL SHELL

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ABSTRACT

It is shown that in an imploding spherical shell the surface instabilities are of two different types. The first, which occurs at the outer surface, is the Rayleigh-Taylor instability. The second instability occurs at the inner surface. This latter instability is not as disruptive as R-T modes, but it has three basic properties which differ considerably from those of the R-T instability: (1) it is oscillatory at early times; (2) it grows faster in the long wavelength modes; (3) it depends on the equation of state. It is further shown that this new instability is driven by amplified sound waves in the shell.

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BASIC EQUATIONS

We consider an imploding spherical shell that obeys the following ideal fluid equations:

$$\rho \left(\frac{d\mathbf{v}}{dt} \right) = -\nabla p , \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 , \quad (2)$$

$$\frac{d}{dt} (p/\rho^\gamma) = 0 \quad (3)$$

A self-consistent description of the shell motion can be obtained by introducing Sedov's hypothesis⁸ of self-similar motion, which in the Lagrangian representation is simply given as

$$R(r_0, t) = r_0 f(t) . \quad (4)$$

$$\rho(r_0, t) = \rho_0(r_0) f^{-3}, \quad p(r_0, t) = p_0(r_0) f^{-3\gamma} . \quad (5)$$

THE UNPERTURBED MOTION

Using the time-dependent pressure in Eq. (1), we obtain the unperturbed motion

$$\ddot{f}(t)f^{3\gamma-2}(t) = -\frac{1}{\rho_0 r} \frac{d}{dr}(\rho_0^\gamma(r)) = -\frac{1}{t_c^2} = -1, \quad (6)$$

The time-dependent part of Eq. (6) yields on integration

$$\dot{f} = \left(\frac{2}{\alpha} (f^{-\alpha} - 1)\right)^{1/2}, \quad (7)$$

where the initial values $f(0) = 1$ and $\dot{f}(0) = 0$ have been used. Here $\alpha = 3(\gamma - 1)$. When $\gamma = 5/3$, Eq. (7) can be integrated once again to give

$$f^2 = 1 - t^2, \quad (8)$$

where $0 < t < 1$.

$$\rho_0(r) = \hat{\rho}_0 \left((r^2 - r_-^2) / (r_+^2 - r_-^2) \right)^{1/(\gamma - 1)}, \quad (9)$$

where r_+ and r_- are the outer and inner radii of the shell;

$$p_0(r) = \hat{p}_0 \left((r^2 - r_-^2) / (r_+^2 - r_-^2) \right)^{\gamma / (\gamma - 1)}, \quad (10)$$

where \hat{p}_0 is a constant pressure at $r = r_+$.

STABILITY ANALYSIS

When a perturbation ξ is introduced, the position vector of a fluid element in Lagrangian variables becomes $\underline{R} + \xi$. A straightforward, though somewhat tedious, calculation^{7,11} shows that ξ obeys the equation

$$f^{(3\gamma-1)} \ddot{\xi} = \frac{(\gamma-1)}{2} (r^2 - r_-^2) \nabla \sigma + (\gamma-1) \sigma \underline{r} + \underline{r} \times \underline{\omega} + (\underline{r} \cdot \nabla) \xi, \quad (11)$$

where $\sigma = \nabla \cdot \xi$ and $\underline{\omega} = \nabla \times \xi$.

We now limit the discussion to the case of incompressible, irrotational perturbations for which the surface instabilities occur. That is, we choose the perturbations such that $\nabla \cdot \xi = 0$ and $\nabla \times \xi = 0$. This implies that $\xi = \nabla \chi$ with $\nabla^2 \chi = 0$. In spherical coordinates

$$\chi(r, \theta, \phi) = \sum_{\ell m} [Q_+^{\ell}(t) r^{\ell} + Q_-^{\ell}(t) r^{-(\ell+1)}] Y_{\ell m}(\theta, \phi). \quad (12)$$

To obtain $Q_{\pm}^{\ell}(t)$, we expand the perturbations in spherical harmonics¹¹

$$\xi = \sum_{\ell m} [\xi_1^{\ell m}(r, t) \hat{a}_1 + \xi_2^{\ell m}(r, t) \hat{a}_2 + \xi_3^{\ell m}(r, t) \hat{a}_3], \quad (13)$$

$$\hat{a}_1 = \hat{e}_r Y_{\ell m}, \quad \hat{a}_2 = r \nabla Y_{\ell m}, \quad \hat{a}_3 = \underline{r} \times \nabla Y_{\ell m}. \quad (14)$$

TIME-DEPENDENT EQUATION

The Classical Rayleigh-Taylor Instability:

$$g(t) = \ddot{R}(t) = r \ddot{f}(t) = -r f^{-3\gamma+2}, \quad (16)$$

Eq. (15) can be written as

$$\ddot{Q}_{\pm}^{\ell}(t) + \left[\frac{3}{2} \mp \left(\ell + \frac{1}{2} \right) \right] \frac{|g(t)|}{R(t)} Q_{\pm}^{\ell}(t) = 0, \quad (17)$$

Notice that $Q_{-}^{\ell}(t)$ modes localized at the inner surface are oscillatory in the static limit, whereas $Q_{+}^{\ell}(t)$ modes are unstable.

The growth rate of the unstable mode is given by

$$\gamma_{t}^2 = |g_0| \left(\frac{m}{R_0} \right) = |z_0| k_{\theta}, \quad (18)$$

This is the growth rate of the classical Rayleigh-Taylor instability for static media.

SOLUTIONS FOR AN ARBITRARY TIME

$$Q_{\pm}^{\ell}(t) = c_1 \mathcal{F}_{\pm}(\ell, \gamma, t) + c_2 \mathcal{G}_{\pm}(\ell, \gamma, t) . \quad (19)$$

Here

$$\mathcal{F}_{\pm}(\ell, \gamma, t) = {}_2F_1 \left[\frac{1}{4} + \frac{2+1d_{\pm}}{4\alpha}, \frac{1}{4} + \frac{2-1d_{\pm}}{4\alpha}, \frac{1}{2}; 1 - f^{-\alpha} \right] \quad (20)$$

and

$$\mathcal{G}_{\pm}(\ell, \gamma, t) = (1 - f^{-\alpha})^{1/2} {}_2F_1 \left[\frac{3}{4} + \frac{2+1d_{\pm}}{4\alpha}, \frac{3}{4} + \frac{2-1d_{\pm}}{4\alpha}, \frac{3}{2}; 1 - f^{-\alpha} \right], \quad (21)$$

$$\text{where } d_{\pm} = \left[8\alpha \left(\frac{3}{2} \mp \left(\ell + \frac{1}{2} \right) \right) - (\alpha + 2)^2 \right]^{1/2} .$$

As in Ref. 10, we obtain the stability criteria by evaluating the asymptotic limits:

$$\lim_{t \rightarrow 1} \frac{g_{\pm}(\ell, \gamma, t)}{f(t)} = a_{\pm}^{\ell} f^{(\alpha-2+1d_{\pm})/2} + c. c. \quad (22)$$

and

$$\lim_{t \rightarrow 1} \frac{g_{\pm}(\ell, \gamma, t)}{f(t)} = b_{\pm}^{\ell} f^{(\alpha-2+1d_{\pm})/2} + c. c. , \quad (23)$$

where

$$a_{\pm}^{\ell} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{-1d_{\pm}}{2\alpha})}{\Gamma(\frac{1}{4} + \frac{2-1d_{\pm}}{4\alpha})\Gamma(\frac{1}{4} - \frac{2+1d_{\pm}}{4\alpha})}, \quad b_{\pm}^{\ell} = 1 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{-1d_{\pm}}{2\alpha})}{\Gamma(\frac{3}{4} + \frac{2-1d_{\pm}}{4\alpha})\Gamma(\frac{3}{4} - \frac{2+1d_{\pm}}{4\alpha})}. \quad (24)$$

SUMMARY OF THE ANALYSIS

1) $l = 0$ mode.

For $\gamma < 5/3$, the limits of \mathcal{F}_+/f and \mathcal{G}_+/f are finite.

If $\gamma < 5/3$, \mathcal{F}_-/f and \mathcal{G}_-/f diverge asymptotically, which signals instability.

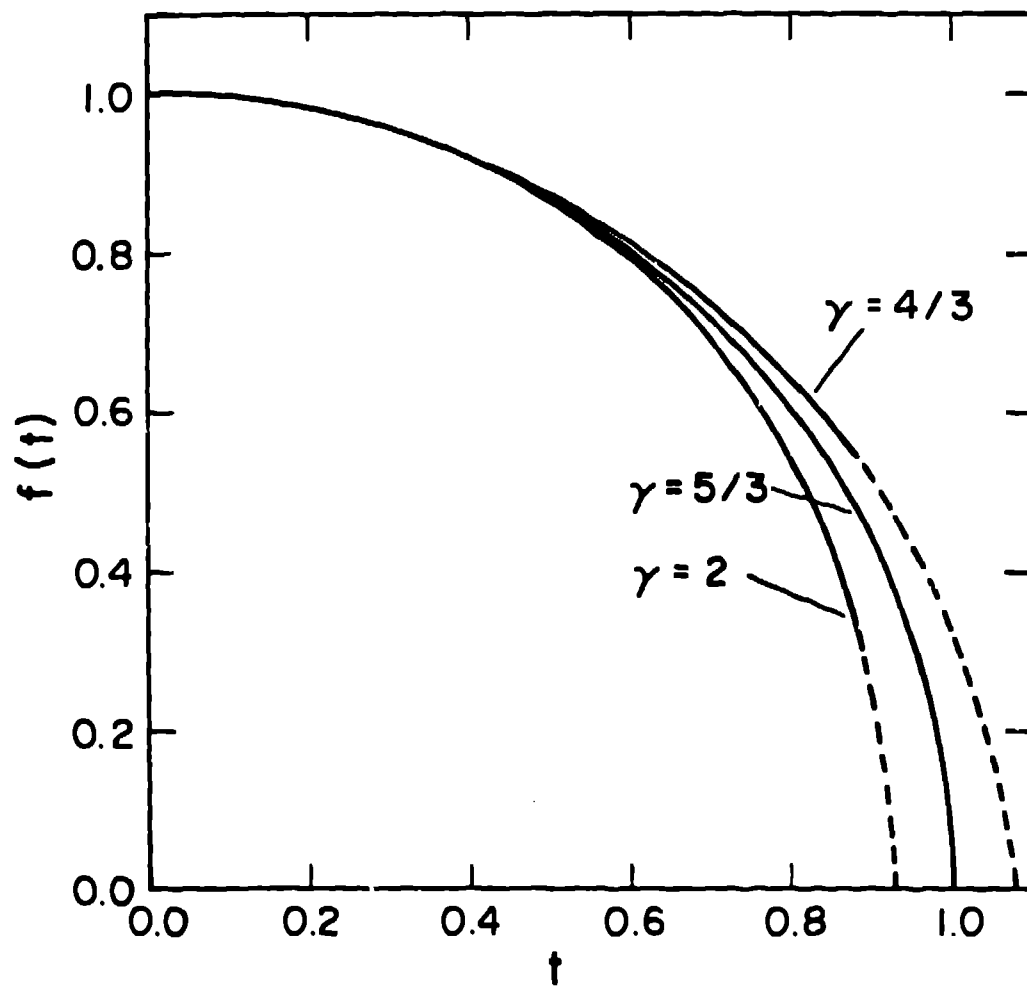
2) The Q_+^l modes with $l > 1$.

In the asymptotic limit, both \mathcal{F}_+/f and \mathcal{G}_+/f diverge as f^κ as $f \rightarrow 0$, where κ is real and negative.

3) The asymptotic limits of both \mathcal{F}_-/f and \mathcal{G}_-/f of Q_-^l diverge for $l > 1$ and $\gamma < 5/3$.

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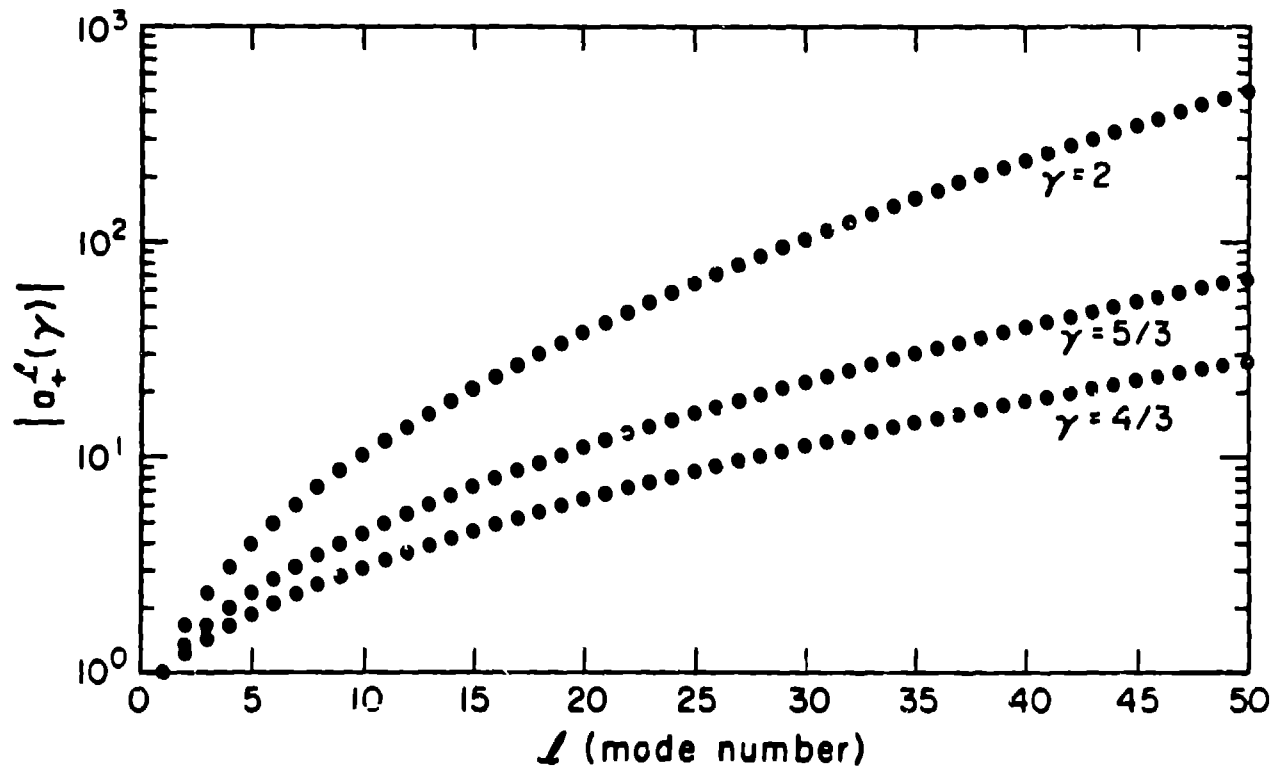


Fig. 1. The absolute values of $a_+^l(\gamma)$ given in Eq. (20) are plotted against the mode number $l = 1$ to 50.

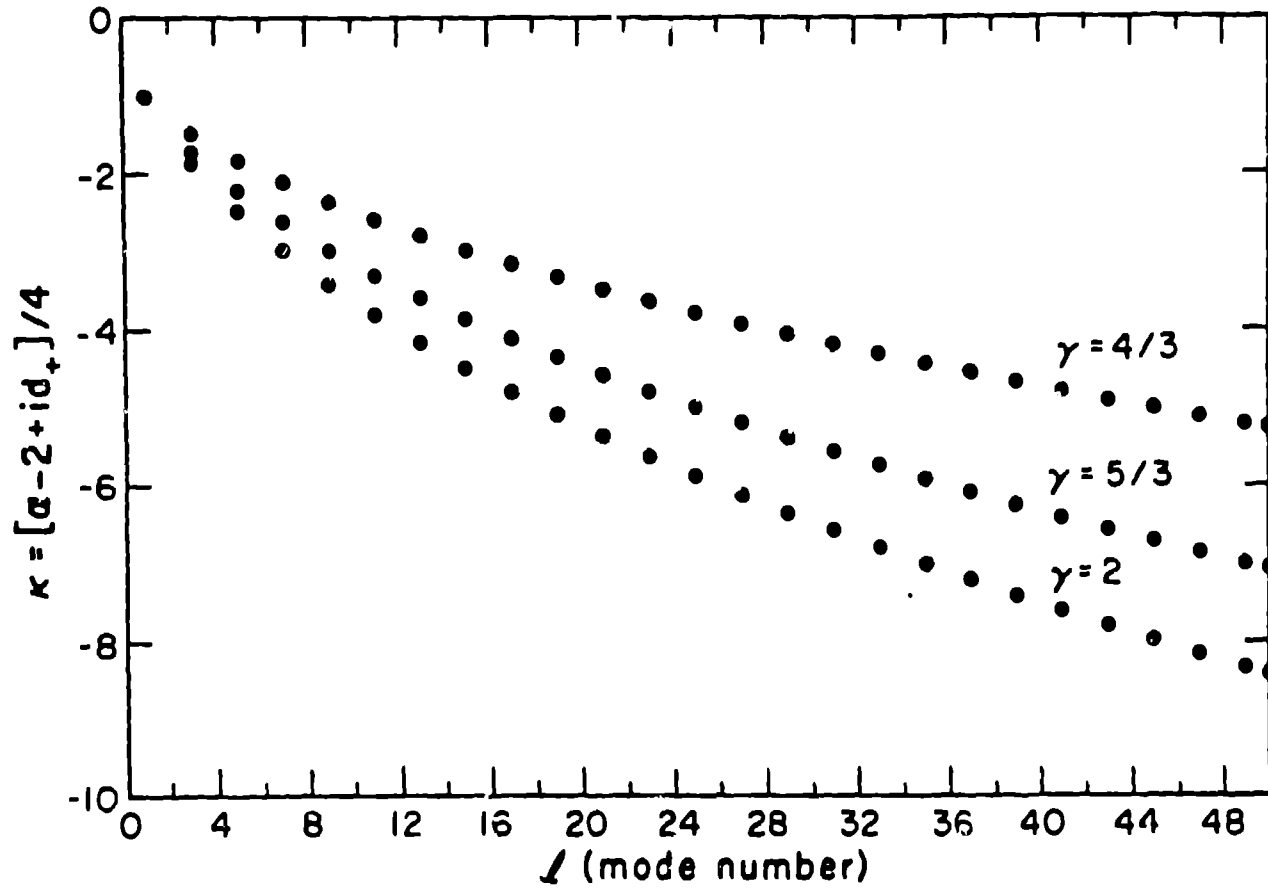


Fig. 2. The exponent of the divergence factor κ is plotted against the mode number.

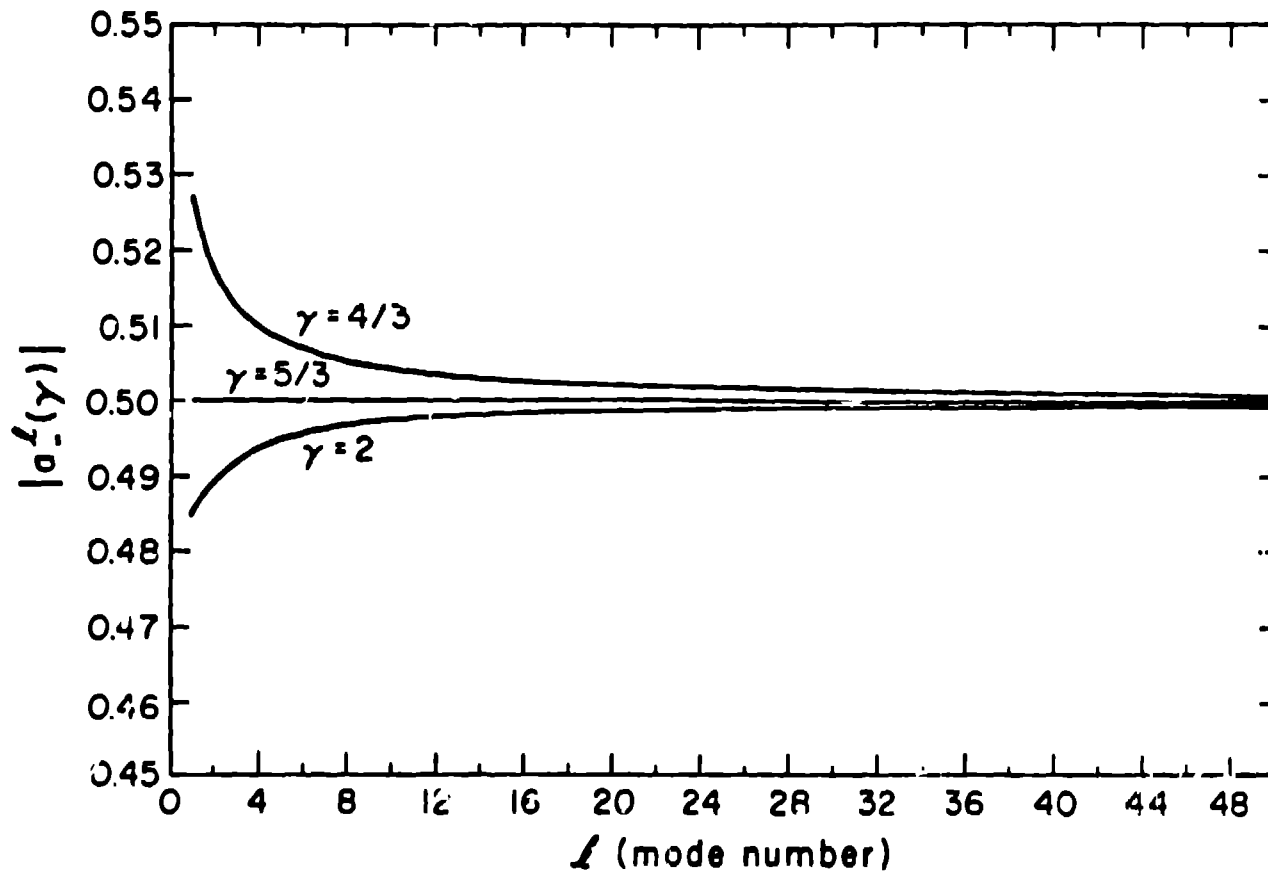


Fig. 3. The absolute values of a_l^γ for $\gamma = 4/3, 5/3,$ and 2 are plotted against the mode number $l = 1$ to 50 .

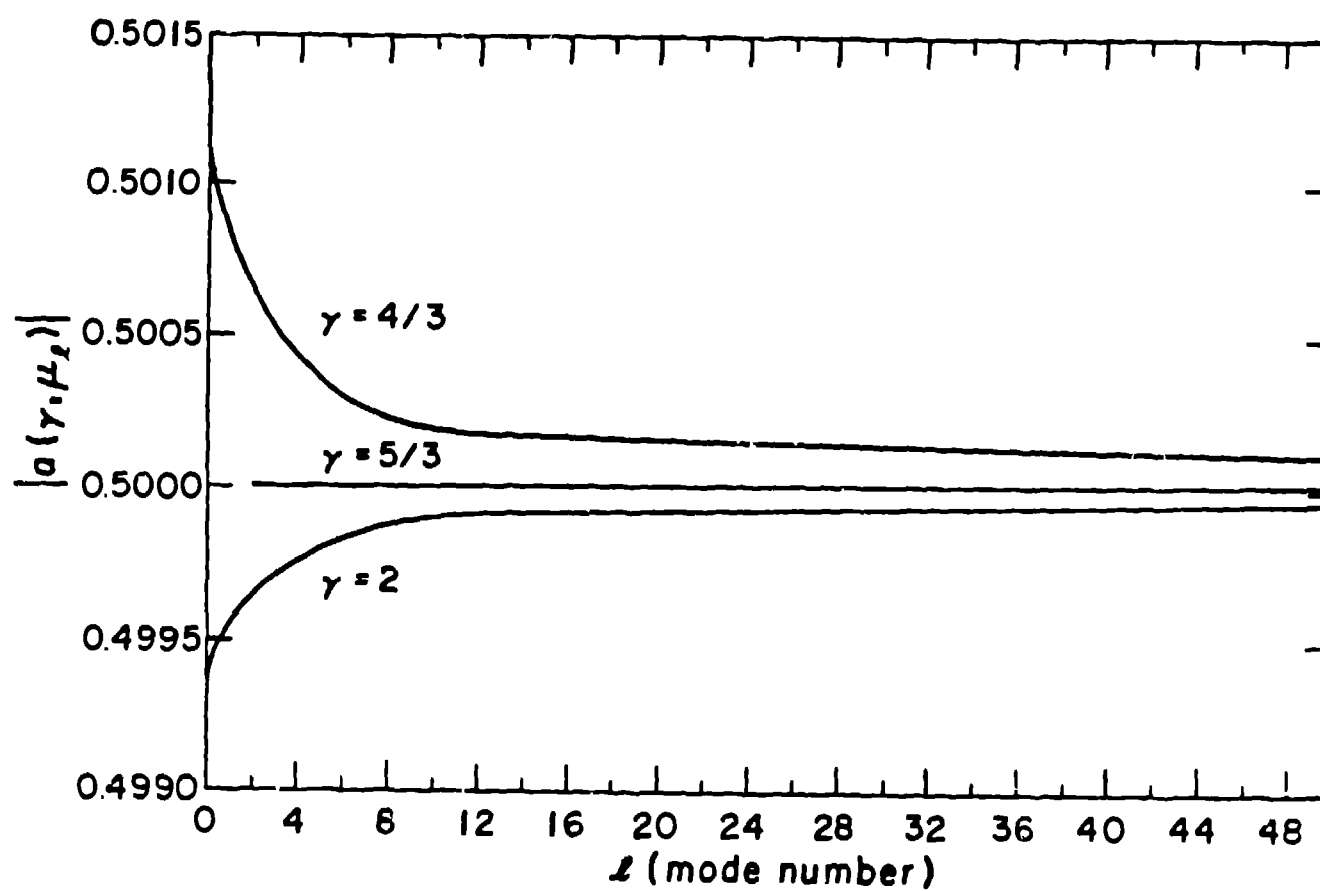


Fig. 4. The absolute values of $a(\gamma, \mu_\ell)$ for $\gamma = 4/3, 5/3,$ and 2 are plotted against the mode number ℓ . Here μ_ℓ is the smallest eigenvalue for a given ℓ with $r_+ = 7.0$ cm and $r_- = 5.0$ cm.