# Fundamental Constants and the

Let us examine Riabouchinsky's paradox a little more carefully and show how its resolution is related to choosing a system of units where the "fundamental constants" (such as Planck's constant $h$ and the speed of light $c$) can be set equal to unity.

The paradox had to do with whether temperature could be used as an independent dimensional unit even though it can be defined as the mean kinetic energy of the molecular motion. Rayleigh had chosen five physical variables (length $l$, temperature difference $\theta$, velocity $v$, specific heat $C$, and heat conductivity $K$) to describe Boussinesq's problem and had assumed that there were four independent dimensions (energy $E$, length $L$, time $T$, and temperature $\Theta$). Thus the solution for $T/T_0$ necessarily is an arbitrary function of *one* dimensionless combination. To see this explicitly, let us examine the dimensions of the five physical variables:

$$[l] = L, [\theta] = \Theta, [v] = LT^{-1}, [C] = EL^{-3}\Theta^{-1},$$

$$\text{and } [K] = EL^{-1}T^{-1}\Theta^{-1} .$$

Clearly the combination chosen by Rayleigh, $lvC/K$, is dimensionless. Although other dimensionless combinations can be formed, they are not independent of the two combinations ($lvC/K$ and $T/T_o$) selected by Rayleigh.

Now suppose, along with Riabouchinsky, we use our knowledge of the kinetic theory to define temperature "as the mean kinetic energy of the molecules" so that $\Theta$ is no longer an independent dimension. This means there are now only *three* independent dimensions and the solution will depend on an arbitrary function of *two* dimensionless combinations. With $\Theta \propto E$, the dimensions of the physical variables become:

$$[l] = L, [\theta] = E, [v] = LT^{-1}, [C] = L^{-3}, \text{and } [K] = L^{-1}T^{-1}.$$

It is clear that, in addition to Rayleigh's dimensionless variable, there is now a new *independent* combination, $Cl^3$ for example, that is dimensionless. To reiterate Rayleigh: "it would indeed be a paradox if the *further* knowledge of the nature of heat . . . put us in a *worse* position than before . . . it would be well worthy of discussion."

Like almost all paradoxes, there is a bogus aspect to the argument. It is certainly true that the kinetic theory allows one to express an energy as a temperature. However, this is only useful and *appropriate* for situations where the physics is dominated by molecular considerations. For macroscopic situations such as Boussinesq's problem, the molecular nature of the system is irrelevant; the microscopic variables have been replaced by macroscopic averages embodied in phenomenological properties such as the specific heat and conductivity. To make Riabouchinsky's identification of energy with temperature is to introduce *irrelevant* physics into the problem.

Exploring this further, we recall that such an energy-temperature identification implicitly involves the introduction of Boltzmann's factor $k$. By its very nature, $k$ will only play an explicit role in a physical problem that directly involves the molecular nature of the system; otherwise it will not enter. Thus one could describe the system from the molecular viewpoint (so that $k$ is involved) and then take a macroscopic limit. Taking the limit is equivalent to setting $k = 0$; the presence of a finite $k$ indicates that explicit effects due to the kinetic theory are important.

With this in mind, we can return to Boussinesq's problem and derive Riabouchinsky's result in a somewhat more illuminating fashion. Let us follow Rayleigh and keep $E$, $L$, $T$, and $\Theta$ as the

Each of these variables, including $F$ itself, is always expressible in terms of some standard set of independent units, which can be chosen to be mass $M$, length $L$, and time $T$. These are the hidden scale parameters. Obviously, other combinations could be used. There could even be other independent units, such as temperature (but remember Riabouchinsky!), or more than one independent length (say, transverse and longitudinal). In this discussion, we shall simply use the conventional $M$, $L$, and $T$. Any generalization is straightforward.

In terms of this standard set of units, the magnitude of each $x_i$ is given by

$$x_i = M^{\alpha_i} L^{\beta_i} T^{\gamma_i} \tag{15}$$

The numbers $\alpha_i$, $\beta_i$, and $\gamma_i$ will be recognized as "the dimensions" of $x_i$. Now suppose we change the system of units by some scale transformation of the form

$$M \to M' = \lambda_M M ,$$

$$L \to L' = \lambda_L L ,$$

and

$$T \to T' = \lambda_T T . \tag{16}$$

Each variable then responds as follows:

$$x_i \to x_i' = Z_i(\lambda) x_i , \tag{17}$$

where

$$Z_i(\lambda) = \lambda_M^{\alpha_i} \lambda_L^{\beta_i} \lambda_T^{\gamma_i} . \tag{18}$$

and $\lambda$ is shorthand for $\lambda_M$, $\lambda_L$, and $\lambda_T$. Since $F$ is itself a dimensional physical quantity, it transforms in an identical fashion under this scale change:

$$F \to F' = Z(\lambda)\, F(x_1, x_2, \ldots, x_n) , \tag{19}$$

where

$$Z(\lambda) = \lambda_M^{\alpha} \lambda_L^{\beta} \lambda_T^{\gamma} . \tag{20}$$

Here $\alpha$, $\beta$, and $\gamma$ are the dimensions of $F$.

There is, however, an alternate but equivalent way to transform from $F$ to $F'$, namely, by transforming each of the variables $x_i$ separately. Explicitly we therefore also have

# Rayleigh-Riabouchinsky Paradox

independent dimensions but add $k$ (with dimensions $E\Theta^{-1}$) as a new physical variable. The solution will now be an arbitrary function of *two* independent dimensionless variables: $lvC/K$ and $kCl^3$. When Riabouchinsky chose to make $Cl^3$ his other dimensionless variable, he, in effect, chose a system of units where $k = 1$. But that was a terrible thing to do here since the physics dictates that $k = 0$! Indeed, if $k = 0$ we regain Rayleigh's original result, that is, we have only *one* dimensionless variable. It is somewhat ironic that Rayleigh's remarks miss the point: "further knowledge of the nature of heat afforded by molecular theory" does not put one in a better position for solving the problem—rather, it leads to a microscopic description of $K$ and $C$. The important point pertinent to the problem set up by Rayleigh is that knowledge of the molecular theory is irrelevant and $k$ must not enter.

The lesson here is an important one because it illustrates the role played by the fundamental constants. Consider Planck's constant $\hbar \equiv h/2\pi$: it would be completely inappropriate to introduce it into a problem of classical dynamics. For example, any solution of the scattering of two billiard balls will depend on macroscopic variables such as the masses, velocities, friction coefficients, and so on. Since billiard balls are made of protons, it might be tempting to the purist to include as a dependent variable the proton-proton total cross section, which, of course, involves $\hbar$. This would clearly be totally inappropriate but is analogous to what Riabouchinsky did in Boussinesq's problem.

Obviously, if the scattering is between two microscopic "atomic billiard balls" then $\hbar$ *must be* included. In this case it is not only quite legitimate but often convenient to choose a system of units where $\hbar = 1$. However, having done so one cannot directly recover the classical limit corresponding to $\hbar = 0$. With $\hbar = 1$, one is stuck in quantum mechanics just as, with $k = 1$, one is stuck in kinetic theory.

A similar situation obviously occurs in relativity: the velocity of light $c$ must not occur in the classical Newtonian limit. However, in a relativistic situation one is quite at liberty to choose units where $c = 1$. Making that choice, though, presumes the physics involves relativity.

The core of particle physics, relativistic quantum field theory, is a synthesis of quantum mechanics and relativity. For this reason, particle physicists find that a system of units in which $\hbar = c = 1$ is not only convenient but is a manifesto that quantum mechanics and relativity are the basic physical laws governing their area of physics. In quantum mechanics, momentum $p$ and wavelength $\lambda$ are related by the de Broglie relation: $p = 2\pi\hbar/\lambda$; similarly, energy $E$ and frequency $\omega$ are related by Planck's formula: $E = \hbar\omega$. In relativity we have the famous Einstein relation: $E = mc^2$. Obviously if we choose $\hbar = c = 1$, all energies, masses, and momenta have the same units (for example, electron volts ($eV$)), and these are the same as *inverse* lengths and times. Thus larger energies and momenta *inevitably* correspond to shorter times and lengths.

Using this choice of units automatically incorporates the profound physics of the uncertainty principle: to probe short space-time intervals one needs large energies. A useful number to remember is that $10^{-13}$ centimeter, or 1 fermi (fm), equals the reciprocal of 200 MeV. We then find that the electron mass ($\approx 1/2$ MeV) corresponds to a length of $\approx 400$ fm—its Compton wavelength. Or the 20 TeV ($2 \times 10^7$ MeV) typically proposed for a possible future facility corresponds to a length of $10^{-18}$ centimeter. This is the scale distance that such a machine will probe! ■

$$F \rightarrow F' = F(Z_1(\lambda)x_1, Z_2(\lambda)x_2, \ldots, Z_n(\lambda)x_n)\,. \qquad (21)$$

Equating these two different ways of effecting a scale change leads to the identity

$$F(Z_1(\lambda)x_1, Z_2(\lambda)x_2, \ldots, Z_n(\lambda)x_n) = Z(\lambda)\, F(x_1, x_2, \ldots, x_n)\,. \qquad (22)$$

As a concrete example, consider the equation $E = mc^2$. To change scale one can either transform $E$ directly or transform $m$ and $c$ separately and multiply the results appropriately—obviously the final result must be the same.

We now want to ensure that the resulting form of the equation does not depend on $\lambda$. This is best accomplished using Euler's trick of taking $\partial/\partial\lambda$ and then setting $\lambda = 1$. For example, if we were to consider changes in the mass scale, we would use $\partial/\partial\lambda_M$ and the chain rule for partial differentiation to arrive at

$$\sum_{i=1}^{n} x_i \frac{\partial Z_i}{\partial \lambda_M} \frac{\partial F}{\partial x_i'} = \frac{\partial Z}{\partial \lambda_M} F\,. \qquad (23)$$

When we set $\lambda_M = 1$, differentiation of Eqs. 18 and 20 yields

$$\left(\frac{\partial Z_i}{\partial \lambda_M}\right)_{\lambda_M=1} = \alpha_i\,,$$

$$\left(\frac{\partial Z}{\partial \lambda_M}\right)_{\lambda_M=1} = \alpha\,. \qquad (24)$$

and $x_i' = x_i$, so that Eq. 23 reduces to

$$\alpha_1 x_1 \frac{\partial F}{\partial x_1} + \alpha_2 x_2 \frac{\partial F}{\partial x_2} + \alpha_3 x_3 \frac{\partial F}{\partial x_3} + \ldots + \alpha_n x_n \frac{\partial F}{\partial x_n} = \alpha F\,. \qquad (25)$$

Obviously this can be repeated with $\lambda_L$ and $\lambda_T$ to obtain a set of three coupled partial differential equations expressing the fundamental *scale invariance of physical laws (that is, the invariance of the physics to the choice of units)* implicit in Fourier's original work. These equations can be solved without too much difficulty; their solution is, in fact, a special case of the solution to the re-