

LA-UR-94-2684

Conf-9408127--2

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

**TITLE: HYDRODYNAMIC INSTABILITIES IN INERTIAL FUSION**

**AUTHOR(S): Nelson M. Hoffman**

**SUBMITTED TO: 45th Scottish Universities Summer School in Physics, St. Andrews, Scotland, UK,  
7 - 20 August 1994**

**DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

**By acceptance of this article, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or to allow others to do so, for U.S. Government purposes.**

**The Los Alamos National Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy.**

**MASTER**  
**Los Alamos** **Los Alamos National Laboratory**  
**Los Alamos, New Mexico 87545**

# Hydrodynamic Instabilities in Inertial Fusion

Nelson M. Hoffman

## Introduction and Survey

An ideal inertial-confinement-fusion (ICF) implosion is exactly spherically symmetric. If the implosion departs from spherical symmetry, the imploding capsule's performance is degraded in several ways: the conversion of the imploding shell's kinetic energy to the fuel's internal energy may be less efficient, the compression of the fuel to high density may be less extreme, and the surface area through which the fuel loses energy by thermal conduction may be increased. In severe cases, asymmetry can lead to the breakup of the imploding shell (at larger spatial scales) or the creation of hydrodynamic turbulence (at smaller spatial scales). Turbulence in turn may have a number of deleterious effects, involving the turbulent transport of mass, momentum, and energy in ways that corrupt the highly organized evolving structure of the imploding capsule.

ICF implosions, whether real or ideal, are subject to a variety of hydrodynamic instabilities that amplify small departures from spherical symmetry. Instabilities can cause a disturbance to grow from an amplitude which may at first seem insignificant to a level that can seriously disrupt the flow, as described above. Instabilities do not themselves generate the initial asymmetric disturbance, or "seed", from which the final disruption grows. Instead, the seeds arise from limitations in our ability to fabricate perfectly spherical shells, to generate perfectly uniform laser beams, or to create perfectly symmetric thermal radiation fields in hohlraums. Small perturbations of a capsule's surface caused by the roughness of the material's crystal structure, or by machining marks from the fabrication process, are examples of instability seeds. Other examples include the interference pattern in a focused laser spot, which can imprint disturbances on an initially smooth surface irradiated by the laser. Thus the seeds simply reflect the inevitable deviation of real-world experiments from the idealized constructs of theory. Instabilities then cause these seeds to grow to a size that may have serious consequences for an ICF implosion.

Hydrodynamic instabilities are straightforward consequences of the conservation equations of hydrodynamics. They are in fact nothing more than solutions to these equations for specific initial and boundary conditions corresponding to somewhat simplified versions of real flow fields. For example, the Rayleigh-Taylor instability, which we shall encounter often in ICF in a generalized form, arises in the case of two initially motionless incompressible fluid layers of unequal density, where the denser fluid is supported atop the less dense fluid in a gravitational field. If the interface, or contact surface, between the layers is disturbed so as not to be exactly horizontal, then the Rayleigh-Taylor instability ensues. The interface disturbance, which is the initial seed in this case, grows until eventually bubbles of the less dense fluid ascend through the denser fluid while jets or "spikes" of the denser fluid plunge downward through the less dense fluid. The Rayleigh-Taylor instability is never encountered in this precise form in ICF, because gravitation plays no role in an ICF implosion; the time and space scales of ICF are simply too small. However, the accelerating and decelerating forces produced by pressure gradients acting on the shell of an ICF

capsule are effectively analogous to gravity. Thus hydrodynamic phenomena arise which are for all practical purposes equivalent to the Rayleigh-Taylor instability, appropriately generalized for compressible flow fields which are converging or diverging, which do not necessarily have sharp boundaries separating fluids of different density, and in which the acceleration force is not necessarily constant in time.

A related instability is the ablation-surface instability, sometimes called "Rayleigh-Taylor instability at an ablation surface." This occurs when intense radiation, either laser or thermal, heats a material interface and ablates it. The ablated material flows away from the interface, creating a high-pressure, low-density corona which accelerates the unablated material. The density decrease from unablated to ablated material corresponds to the contact surface between the two fluids in the incompressible Rayleigh-Taylor instability, while the pressure increase from unablated to ablated material gives an acceleration force.

Other instabilities of particular consequence for ICF are the Richtmyer-Meshkov instability and the Kelvin-Helmholtz instability. The Richtmyer-Meshkov instability occurs when a shock wave crosses the interface between two fluids of unequal density, traveling in a direction normal to the interface. This can be viewed as the limiting case of a Rayleigh-Taylor instability in which gravity acts for an infinitesimally short duration on the fluids, imparting an impulsive acceleration to the interface and generating fluid motions that persist even in the absence of gravity. The Kelvin-Helmholtz instability arises when the two fluids are initially in motion, and there is a variation across the interface of the velocity component parallel to the interface. Such a gradient in the parallel velocity, referred to as "velocity shear", is unstable if it is sufficiently severe, and leads to the creation of vortices which entrain the two fluids in a characteristic whirling rotational motion. The Kelvin-Helmholtz instability can arise along the interface between the ascending bubbles and descending spikes of late-stage Rayleigh-Taylor instability, where the shear due to the differential motion of these structures can be significant. Other circumstances giving rise to Kelvin-Helmholtz instability include the case of a shock wave crossing an interface in a direction not precisely normal to the interface. Shear is generated by the deviation from strict normality, so that both Kelvin-Helmholtz and Richtmyer-Meshkov instability can occur in this situation.

**The occurrence of instabilities in an ICF implosion**  
**Ablation-surface instability during ablative acceleration**  
**Richtmyer-Meshkov, Kelvin-Helmholtz during shock emergence**  
**Rayleigh-Taylor during deceleration**

**A. Linear analysis of Rayleigh-Taylor Instability**

It is worthwhile to derive from basic principles the small-amplitude behavior of the Rayleigh-Taylor instability, both because we shall thereby discover some of the properties of the instability and because the exercise will serve as an example of the technique of linear perturbation analysis, widely used in instability studies. Our starting point is the system of equations describing the hydrodynamic motion of an ideal fluid (that is, a fluid in which there is no energy dissipation or heat exchange), known as the Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \rho \mathbf{g}. \quad (2)$$

Equation (1) is called the *continuity equation* and Eq. (2) is called the *equation of motion* or *momentum equation*. Here  $\rho, \mathbf{v} \equiv v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$ , and  $p$  denote respectively the density, velocity, and pressure of the fluid. An external force, such as gravity, acting on the fluid is represented by  $\mathbf{g} \equiv g_x \hat{x} + g_y \hat{y} + g_z \hat{z}$ . In the particular example of Rayleigh-Taylor instability we shall consider, the fluids meet at a horizontal interface and are initially at rest. We take the normal to the interface as the direction  $\hat{z}$ , so that gravity acts along  $\hat{z}$ :  $\mathbf{g} = g_z \hat{z}$ . Since gravity acts downward,  $g_z < 0$  and  $\mathbf{g} = -|g_z| \hat{z}$ . All physical quantities are initially uniform throughout both fluids, away from the interface.

To investigate the stability of hydrodynamic motion, we ask how the motion responds to a small fluctuation in the value of any of the flow variables appearing in the Euler equations. If the fluctuation grows in amplitude so that the flow never returns to its initial state, we say that the flow is *unstable* with respect to fluctuations of that type. Accordingly, we replace the variables in Eqs. (1) and (2) as follows:

$$\rho = \rho_0 + \rho_1,$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1,$$

$$p = p_0 + p_1.$$

The quantities with subscripts "0" represent the unperturbed, or "zeroth-order" motion of the fluid, and thus must themselves satisfy Eqs. (1) and (2). The quantities with subscripts "1" represent a small perturbation about the zeroth-order quantities; that is,  $\rho_1 \ll \rho_0$ ,  $\mathbf{v}_1 \ll \mathbf{v}_0$ , and  $p_1 \ll p_0$ . Substituting these expressions into Eqs. (1) and (2) gives

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \nabla \cdot [(\rho_0 + \rho_1)(\mathbf{v}_0 + \mathbf{v}_1)] = 0,$$

$$(\rho_0 + \rho_1) \frac{\partial(\mathbf{v}_0 + \mathbf{v}_1)}{\partial t} + (\rho_0 + \rho_1)[(\mathbf{v}_0 + \mathbf{v}_1) \cdot \nabla](\mathbf{v}_0 + \mathbf{v}_1) = -\nabla(p_0 + p_1) + (\rho_0 + \rho_1)\mathbf{g},$$

or

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_0 + \rho_1 \mathbf{v}_0 + \rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_1) = 0, \quad (3)$$

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{v}_0}{\partial t} + \rho_1 \frac{\partial \mathbf{v}_0}{\partial t} + \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_1 \frac{\partial \mathbf{v}_1}{\partial t} + \\ \rho_0(\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1) + \rho_1(\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1) = \\ -\nabla p_0 - \nabla p_1 + \rho_0 \mathbf{g} + \rho_1 \mathbf{g}. \end{aligned} \quad (4)$$

The fact that the zeroth-order quantities satisfy Eqs. (1) and (2) means

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_0) = 0, \quad (5)$$

$$\rho_0 \frac{\partial \mathbf{v}_0}{\partial t} + \rho_0 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 = -\nabla p_0 + \rho_0 \mathbf{g}. \quad (6)$$

We can subtract the zeroth-order equations Eqs. (5) and (6) from Eqs. (3) and (4). This amounts to dropping all terms in Eqs. (3) and (4) which contain no appearances of the subscript "1". Furthermore, we can omit terms in Eqs. (3) and (4) which contain products of first-order quantities, since they are very small in comparison to terms which are linear in first-order quantities. This process of omission of quadratic quantities, by which we obtain a system of linear partial differential equations, is called *linearization* of the perturbed equations. Linearization is valid only if the perturbations are small. It means in effect that we drop all terms in Eqs. (3) and (4) in which the subscript "1" appears twice. The result of linearizing and of subtracting the zeroth-order equations is that Eqs. (3) and (4) become

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \mathbf{v}_0 + \rho_0 \mathbf{v}_1) = 0, \quad (7)$$

$$\rho_1 \frac{\partial \mathbf{v}_0}{\partial t} + \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_0 (\mathbf{v}_1 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1) + \rho_1 (\mathbf{v}_0 \cdot \nabla \mathbf{v}_0) = -\nabla p_1 + \rho_1 \mathbf{g}. \quad (8)$$

We now restrict our attention to the Rayleigh-Taylor instability in particular. For the problem as it was posed earlier, the fluids are initially at rest. This means that  $\mathbf{v}_0 = 0$ , so that Eqs. (7) and (8) become

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0, \quad (9)$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \rho_1 \mathbf{g}. \quad (10)$$

We now appeal to the fact that, for many situations of interest in ICF, unstable flow occurs at velocities much smaller than the local sound speed. This has the effect that accelerations in the flow are not strong enough to change the density of a fluid element significantly, so the fluid moves without compressing or expanding. In such a situation we call the flow *incompressible*. Provided that we are well away from shock waves or centers of convergence, the assumption of incompressible flow is often valid. To say that fluid elements move without changing density is to say that the Lagrangian total derivative of density is zero, that is,

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0. \quad (11)$$

Applying this equation to our instability analysis, we substitute the perturbed expressions  $\rho = \rho_0 + \rho_1$  and  $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$  into Eq. (11), and recall that both  $\mathbf{v}_0$  and the time

derivative of  $\rho_0$  vanish, since they describe the static initial state. We also linearize the result, dropping nonlinear terms in the first-order quantities, as before. The result is that Eq. (11) becomes

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 = 0. \quad (12)$$

Comparing this equation to Eq. (9), which we write in expanded form as

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \rho_0 = 0, \quad (13)$$

we see that subtracting Eq. (12) from Eq. (13) yields

$$\nabla \cdot \mathbf{v}_1 = 0. \quad (14)$$

This is a consequence of the assumption of incompressible flow. We can use either Eq. (12) or Eq. (14) to replace the linearized continuity equation Eq. 9 under this assumption.

To proceed, we write out the vector equations (10) and (12) in component form. The linearized momentum equation (10) becomes

$$\begin{aligned} \rho_0 \frac{\partial v_{1x}}{\partial t} &= -\frac{\partial p_1}{\partial x} + \rho_1 g_x, \\ \rho_0 \frac{\partial v_{1y}}{\partial t} &= -\frac{\partial p_1}{\partial y} + \rho_1 g_y, \\ \rho_0 \frac{\partial v_{1z}}{\partial t} &= -\frac{\partial p_1}{\partial z} + \rho_1 g_z, \end{aligned}$$

while the linearized incompressible continuity equation (12) becomes

$$\frac{\partial \rho_1}{\partial t} + v_{1x} \frac{\partial \rho_0}{\partial x} + v_{1y} \frac{\partial \rho_0}{\partial y} + v_{1z} \frac{\partial \rho_0}{\partial z} = 0.$$

Because gravity acts only in the  $\hat{\mathbf{z}}$  direction,  $g_x = g_y = 0$ . Furthermore, since  $\rho_0$  is uniform throughout each medium, with its only variation occurring across the horizontal interface, we have  $\partial \rho_0 / \partial x = \partial \rho_0 / \partial y = 0$ , while  $\partial \rho_0 / \partial z$  is non-zero, but only at the interface. Thus the linearized incompressible component equations may be written, using  $g = |g_z| = -g_z$ ,

$$\rho_0 \frac{\partial v_{1x}}{\partial t} = -\frac{\partial p_1}{\partial x}, \quad (15)$$

$$\rho_0 \frac{\partial v_{1y}}{\partial t} = -\frac{\partial p_1}{\partial y}, \quad (16)$$

$$\rho_0 \frac{\partial v_{1z}}{\partial t} = -\frac{\partial p_1}{\partial z} - \rho_1 g, \quad (17)$$

$$\frac{\partial \rho_1}{\partial t} + v_{1x} \frac{\partial \rho_0}{\partial x} = 0. \quad (18)$$

It will also be useful to have Eq. (14), which expresses the nondivergence of the first-order flow, in component form:

$$\frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1y}}{\partial y} + \frac{\partial v_{1z}}{\partial z} = 0. \quad (19)$$

The next step in our analysis is to carry out a Fourier transformation of the system of equations (15) - (19). This is a powerful technique for the solution of differential equations, because of a useful property of Fourier transforms: if  $F[f(t)]$  is the Fourier transform of the function  $f(t)$  with respect to the independent variable  $t$ , then the Fourier transform of the derivative  $df/dt$  is

$$F[df/dt] = i\omega F[f(t)],$$

where  $s$  is the transform variable. Thus a differential operator acting on a physical quantity becomes simply a product of the corresponding transform variable and the Fourier transform of that quantity. Accordingly we define the following two-dimensional Fourier transforms with respect to  $x$  and  $y$ :

$$\begin{aligned} V_{1x}(k_x, k_y, z, t) &= F_{xy}[v_{1x}(x, y, z, t)] \\ &\equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy v_{1x}(x, y, z, t) e^{-(k_x x + k_y y)}, \\ V_{1y}(k_x, k_y, z, t) &= F_{xy}[v_{1y}(x, y, z, t)], \\ V_{1z}(k_x, k_y, z, t) &= F_{xy}[v_{1z}(x, y, z, t)], \\ P(k_x, k_y, z, t) &= F_{xy}[p_1(x, y, z, t)], \\ R(k_x, k_y, z, t) &= F_{xy}[\rho_1(x, y, z, t)]. \end{aligned}$$

We do not transform with respect to  $z$ , because the  $\hat{z}$  direction does not share the symmetry of the other two directions; the linearized component equations (15)-(19) are invariant under the interchange of  $x$  and  $y$ , and it will turn out that solutions are waves in the  $(x, y)$  plane. Furthermore, zeroth-order quantities such as  $\rho_0$  are not functions of  $x$  and  $y$ , which simplifies the Fourier integrals. The transform variables  $k_x$  and  $k_y$  are called the  $\hat{x}$  and  $\hat{y}$  components, respectively, of the *wavevector*  $k$ , whose magnitude  $k = \sqrt{k_x^2 + k_y^2}$  is called the *wavenumber*.

Additionally, we shall seek solutions whose time dependence is proportional to  $e^{\gamma t}$ . This is a standard procedure when Fourier transforming differential equations. If, for example, we suppose that

$$V_{1z}(k_x, k_y, z, t) = \tilde{V}_{1z}(k_x, k_y, z) e^{\gamma t},$$

then

$$\frac{\partial V_{1z}}{\partial t} = \gamma \tilde{V}_{1z} e^{\gamma t} = \gamma V_{1z}.$$

Thus, again, a derivative can be replaced by a product. The variable  $\gamma$  is called the *frequency*.

Performing the Fourier transforms of the component equations (15)-(19), and making the assumption that the time dependence of the solution is given by  $e^{\gamma t}$ , where  $\gamma$  may be a function of  $k_x$  and  $k_y$ , results in

$$\gamma \rho_0 V_{1x} = -ik_x P, \quad (20)$$

$$\gamma \rho_0 V_{1y} = -ik_y P, \quad (21)$$

$$\gamma \rho_0 V_{1z} = -\frac{\partial P}{\partial z} - g R, \quad (22)$$

$$\gamma R + V_{1z} \frac{\partial \rho_0}{\partial z} = 0, \quad (23)$$

$$ik_x V_{1x} + ik_y V_{1y} + \frac{\partial V_{1z}}{\partial z} = 0. \quad (24)$$

The solution of the equations (20) - (24) is now straightforward. Multiply Eq. (20) by  $ik_x$  and multiply Eq. (21) by  $ik_y$ :

$$i\gamma \rho_0 k_x V_{1x} = k_x^2 P,$$

$$i\gamma \rho_0 k_y V_{1y} = k_y^2 P.$$

Add these equations:

$$i\gamma \rho_0 (k_x V_{1x} + k_y V_{1y}) = (k_x^2 + k_y^2) P. \quad (25)$$

From Eq. (24), solve for  $k_x V_{1x} + k_y V_{1y} = i\partial V_{1z}/\partial z$  and substitute in Eq. (25):

$$-\gamma \rho_0 \frac{\partial V_{1z}}{\partial z} = k^2 P, \quad (26)$$

using  $k^2 = k_x^2 + k_y^2$ , the square of the wavenumber. Next we eliminate  $R$  between Eqs. (22) and (23). Equation (23) implies

$$R = -\frac{1}{\gamma} V_{1z} \frac{\partial \rho_0}{\partial z}$$

which we may substitute into Eq. (22) to obtain

$$\frac{\partial P}{\partial z} = -\gamma \rho_0 V_{1z} + \frac{g}{\gamma} V_{1z} \frac{\partial \rho_0}{\partial z}.$$

Finally, solving for  $P$  from Eq. (26) and inserting here we find

$$\frac{\partial}{\partial z} \left( \frac{-\gamma \rho_0}{k^2} \frac{\partial V_{1z}}{\partial z} \right) = -\gamma \rho_0 V_{1z} + \frac{g}{\gamma} V_{1z} \frac{\partial \rho_0}{\partial z}$$

or

$$\frac{\partial}{\partial z} \left( \rho_0 \frac{\partial V_{1z}}{\partial z} \right) = k^2 \rho_0 V_{1z} \left( 1 - \frac{g}{\gamma^2 \rho_0} \frac{\partial \rho_0}{\partial z} \right). \quad (27)$$

Everywhere except at the interface,  $\rho_0$  is constant, so its  $z$ -derivative vanishes and  $\rho_0$  may be canceled from Eq. (27), leaving

$$\frac{\partial^2 V_{1z}}{\partial z^2} = k^2 V_{1z}.$$

The general solution to this equation is

$$V_{1z} = Ae^{+kz} + Be^{-kz}.$$

The vertical velocity should vanish at large distances from the interface, and so we choose a solution with  $A \neq 0, B = 0$  for  $z < 0$  and with  $A = 0, B \neq 0$  for  $z > 0$ . In order that  $V_{1z}$  be continuous across the interface, we select

$$V_{1z} = \begin{cases} We^{+kz}, & z < 0 \\ We^{-kz}, & z > 0 \end{cases},$$

where  $W = V_{1z}(z = 0)$  is the value at the interface.

The derivative  $\partial V_{1z}/\partial z$  is not continuous, however. It has the value  $kW$  immediately below the interface and  $-kW$  immediately above. Equation (27) expresses the relationship between the discontinuity in  $\partial V_{1z}/\partial z$  and the discontinuity in density. We can use this relationship, which is essentially a boundary condition at the interface, to determine the frequency  $\gamma$  in terms of the gravity, the wavevector, and the density jump.

To do so, we integrate Eq. (27) over an infinitesimal element of  $z$  that includes the interface  $z = 0$ . The derivative of a quantity, when integrated, then gives simply the change in the value of that quantity across the interface. Thus the left-hand side of Eq. (27) integrates to

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial z} \left( \rho_0 \frac{\partial V_{1z}}{\partial z} \right) dz &= \rho_0 \frac{\partial V_{1z}}{\partial z} \Big|_{-\epsilon}^{\epsilon} = -\rho_0(z > 0)kW - \rho_0(z < 0)kW \\ &= -kW(\rho_{\text{above}} + \rho_{\text{below}}) \equiv I_1, \end{aligned}$$

where  $\rho_{\text{above}} \equiv \rho_0(z > 0)$  is the density in the upper fluid and  $\rho_{\text{below}} \equiv \rho_0(z < 0)$  is the density in the lower fluid. The first term on the right-hand side of Eq. (27) gives, upon integration,

$$\int_{-\epsilon}^{\epsilon} k^2 \rho_0 V_{1z} dz = k^2 W (\rho_{\text{above}} \epsilon + \rho_{\text{below}} \epsilon) \equiv I_2.$$

The second term on the right-hand side of Eq. (27) gives

$$\begin{aligned} - \int_{-\epsilon}^{\epsilon} k^2 \rho_0 V_{1z} \frac{g}{\gamma^2 \rho_0} \frac{\partial \rho_0}{\partial z} dz &= - \frac{k^2 W g}{\gamma^2} \int_{-\epsilon}^{\epsilon} \frac{\partial \rho_0}{\partial z} dz = - \frac{k^2 W g}{\gamma^2} \rho_0 \Big|_{-\epsilon}^{\epsilon} \\ &= - \frac{k^2 W g}{\gamma^2} (\rho_{\text{above}} - \rho_{\text{below}}) \equiv I_3. \end{aligned}$$

In the limit that  $\epsilon$  goes to zero,  $I_2$  vanishes, because it is proportional to  $\epsilon$ . On the other hand,  $I_1$  and  $I_3$  are finite; they are, in effect, integrals of delta functions. Thus we must have  $I_1 = I_3$  or

$$-kW(\rho_{\text{above}} + \rho_{\text{below}}) = -\frac{k^2 W g}{\gamma^2}(\rho_{\text{above}} - \rho_{\text{below}}).$$

Solving for  $\gamma$ , we obtain

$$\gamma^2 = kg \frac{(\rho_{\text{above}} - \rho_{\text{below}})}{(\rho_{\text{above}} + \rho_{\text{below}})}.$$

Define a dimensionless number  $A$ , called the *Atwood number*:

$$A \equiv \frac{(\rho_{\text{above}} - \rho_{\text{below}})}{(\rho_{\text{above}} + \rho_{\text{below}})}.$$

Then  $\gamma^2 = kgA$ . Since solutions depend on time as  $e^{\gamma t}$ , we have, for example,

$$V_{1z} = \begin{cases} \tilde{W}(k_z, k_y) e^{+k_z z} e^{\gamma t}, & z < 0 \\ \tilde{W}(k_z, k_y) e^{-k_z z} e^{\gamma t}, & z > 0 \end{cases}.$$

If  $\rho_{\text{above}} > \rho_{\text{below}}$ , then  $A$  is positive, the interface is unstable, and the perturbation grows exponentially with growth rate  $\gamma = \sqrt{kgA}$ . If, on the other hand,  $\rho_{\text{above}} < \rho_{\text{below}}$ , then  $A$  is negative,  $\gamma$  is imaginary, and the interface oscillates with frequency  $\text{Im}(\gamma) = \sqrt{kgA}$ .

## B. Ablation-Surface Instability

The ablation-surface instability occurs when a material layer is rapidly heated by some energy-deposition process and ablates. If the spatial extent of the energy-deposition region is small with respect to the depth of the layer, then a high-pressure low-density region forms adjacent to the layer, which accelerates the layer. The low-density region is composed of heated ablating material expanding away from the layer's surface. The acceleration of the high-density layer by the low-density ablated material is analogous to the support of a high-density fluid by a low-density fluid in a gravitational field, so an instability arises. This ablation-surface instability is much like the classical Rayleigh-Taylor instability, just discussed, but differs because of the flow of material out of the high-density layer, across the ablation surface, and into the low-density ablated region. Furthermore, gravity plays no role.

If we approximate the energy-deposition region as a discontinuity, we can make a rough estimate (following Gamaly 1993) of the effect of ablation on the growth of perturbations by repeating the Rayleigh-Taylor analysis with a simple change: because of the ablation flow, we permit a velocity discontinuity at the interface as well as a density discontinuity. This means that the zeroth-order state is not static, so that we cannot set  $\mathbf{v}_0 = 0$  in our linear perturbation analysis.

We consider a reference frame moving with the layer. In this frame, the layer is at rest, and the ablating material moves in the  $-\hat{z}$  direction with velocity  $v_{\text{abl}}$ . Thus

$$\mathbf{v}_0 = \begin{cases} -v_{\text{abl}}\hat{z}, & z < 0 \\ 0, & z > 0 \end{cases}.$$

Rewriting Eq. (8), omitting gravity, and keeping terms containing  $\mathbf{v}_0$  gives

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_1 \left( \frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 \right) + \rho_0 (\mathbf{v}_1 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1) = -\nabla p_1. \quad (*a)$$

Rewriting Eq. (6) and omitting gravity gives

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = -\frac{1}{\rho_0} \nabla p_0. \quad (*b)$$

We may substitute this expression for the total Lagrangian derivative of  $\mathbf{v}_0$  into Eq. (\*a) and rearrange terms to obtain

$$\rho_0 \left( \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 \right) = \frac{\rho_1}{\rho_0} \nabla p_0 - \nabla p_1. \quad (*c)$$

Just as in the classical Rayleigh-Taylor analysis, we take the two-dimensional Fourier transform of the  $z$ -component of Eq. (\*c), assuming a time dependence like  $e^{\gamma t}$ . This leads to

$$\rho_0 \left( \gamma V_{1z} + V_{1z} \frac{\partial v_{0z}}{\partial z} + v_{0z} \frac{\partial V_{1z}}{\partial z} \right) = \frac{R}{\rho_0} \frac{\partial p_0}{\partial z} - \frac{\partial P}{\partial z}, \quad (*d)$$

where, as before,  $V_{1z} = F[v_{1z}]$ ,  $R = F[\rho_1]$ , and  $P = F[p_1]$ .

We define an acceleration  $g_0 \equiv -(1/\rho_0) \partial p_0 / \partial z$ , and we use Eq. (23) to eliminate  $R = -(1/\gamma) V_{1z} \partial \rho_0 / \partial z$  in Eq. (\*d). Then solving Eq. (\*d) for  $\partial P / \partial z$  results in

$$\frac{\partial P}{\partial z} = -\gamma \rho_0 V_{1z} + \frac{g_0}{\gamma} V_{1z} \frac{\partial \rho_0}{\partial z} - \rho_0 V_{1z} \frac{\partial v_{0z}}{\partial z} - \rho_0 v_{0z} \frac{\partial V_{1z}}{\partial z}. \quad (*e)$$

Finally, we use Eq. (26) to eliminate  $P$  in terms of  $\partial V_{1z} / \partial z$ , and find upon multiplying by  $-k^2 / \gamma$

$$\frac{\partial}{\partial z} \left( \rho_0 \frac{\partial V_{1z}}{\partial z} \right) = k^2 \rho_0 V_{1z} - g_0 \frac{k^2}{\gamma^2} V_{1z} \frac{\partial \rho_0}{\partial z} + \frac{k^2}{\gamma} \rho_0 V_{1z} \frac{\partial v_{0z}}{\partial z} + \frac{k^2}{\gamma} \rho_0 v_{0z} \frac{\partial V_{1z}}{\partial z}. \quad (*f)$$

Equation (\*f) is analogous to Eq. (27) in the classical Rayleigh-Taylor analysis, but contains two additional terms on the right-hand side, proportional to the zeroth-order velocity and its  $z$ -gradient. Again, we use this equation to derive a jump condition at the interface by integrating it over an infinitesimal element  $-\epsilon \leq z \leq \epsilon$ , which includes the interface. We shall find as usual that only the terms in Eq. (\*f) which are delta functions produce any finite contribution to the integral in the limit that  $\epsilon \rightarrow 0$ . These are the second and third terms on the right-hand side, and the term on the left-hand side;  $\rho_0$ ,  $v_{0z}$ , and  $\partial V_{1z} / \partial z$  are discontinuous at the interface, so their  $z$ -derivatives are delta functions. However,  $\partial V_{1z} / \partial z$  itself is not a delta function, so the fourth term on the right-hand side produces a vanishing integral. So does the first term on the right-hand side, as we saw in Eq. (\*\*b) in the classical Rayleigh-Taylor analysis.

Integrating Eq. (\*f) requires evaluating only one new term, since two of the non-vanishing terms were integrated earlier, as  $I_1$  in Eq. (\*\*a) and  $I_3$  in Eq. (\*\*c). The new integral is

$$\int_{-\epsilon}^{\epsilon} \frac{k^2}{\gamma} \rho_0 V_{1z} \frac{\partial v_{0z}}{\partial z} dz = k^2 \frac{W}{\gamma} \int_{-\epsilon}^{\epsilon} \rho_0 \frac{\partial v_{0z}}{\partial z} dz \equiv I_4,$$

where  $W$  is the value of  $V_{1z}$  at  $z = 0$ . The integral is not so straightforward to evaluate as those encountered earlier, since the integrand is the product of a step function and a delta function. However, let us suppose that  $\rho_0$  and  $v_0$  vary linearly over the infinitesimal region  $-\epsilon \leq z \leq \epsilon$ , so that as  $\epsilon \rightarrow 0$  they approach step functions and  $\partial v_{0z}/\partial z$  approaches a delta function. Then the integral is trivial, with the result that

$$I_4 = k^2 \frac{W}{\gamma} \frac{\rho_{un} + \rho_{abl}}{2} v_{abl},$$

where  $\rho_{un}$  and  $\rho_{abl}$  are the densities in the unablated layer and in the ablated region, respectively.

Thus the result of integrating Eq. (\*f) is

$$I_1 = I_3 + I_4,$$

or

$$-kW(\rho_{un} + \rho_{abl}) = -\frac{k^2 W g_0}{\gamma^2} (\rho_{un} - \rho_{abl}) + \frac{k^2 W}{\gamma} \frac{\rho_{un} + \rho_{abl}}{2} v_{abl},$$

which can be simplified as

$$\gamma^2 + \frac{kv_{abl}}{2}\gamma - kg_0 A = 0, \quad (*g)$$

where

$$A \equiv \frac{(\rho_{un} - \rho_{abl})}{(\rho_{un} + \rho_{abl})}.$$

The solution to the quadratic equation (\*g) for  $\gamma$  is

$$\gamma = -\frac{kv_{abl}}{4} \pm \sqrt{\left(\frac{kv_{abl}}{4}\right)^2 + kg_0 A}.$$

The positive root may be written, when  $\sqrt{kg_0 A} \gg kv_{abl}/4$ ,

$$\gamma = \sqrt{kg_0 A} - \frac{kv_{abl}}{4} + \frac{1}{2\sqrt{kg_0 A}} \left(\frac{kv_{abl}}{4}\right)^2 - \dots \quad (*h)$$

The effect of ablation is thus to reduce the growth rate of the instability.

Although this expression is only approximate, having been derived under some rather severe restrictions (no spatial extent of the region of acceleration, no modification of the continuity equation for finite zeroth-order velocity, no heating or energy exchange), it

nevertheless resembles relations obtained from more accurate treatments. For example, the *Takabe relation* (Takabe et al. 1985)

$$\gamma = \alpha \sqrt{kg_0} - \beta kv_a \quad (*i)$$

is found to describe detailed numerical solutions of a linear perturbation analysis of ablation-surface instability that includes heating and energy exchange in the flow. The analysis results in a system of five coupled ordinary differential equations for first-order variations in five quantities: density, normal velocity, tangential velocity, temperature, and normal heat flux. In general, the solutions are well fit using  $\alpha = 0.9$  and  $3 < \beta < 4$ . In Eq. (\*i), the ablation velocity  $v_a$  denotes the mass ablation rate divided by the density at the ablation surface, whereas  $v_{abl}$  in Eq. (\*h) represents the terminal velocity reached by ablating material far from the ablation surface. We expect  $v_{abl} \gg v_a$ , which accounts in part for the different coefficients of  $kv_{abl}$  and  $kv_a$  in eqs. (\*h) and (\*i).

### C. Bubble rise in late-stage Rayleigh-Taylor instability

The amplitude of a sinusoidal perturbation increases exponentially with time in the early stage of Rayleigh-Taylor instability, as we saw earlier in the linear analysis. Eventually the growth rate decreases, when the amplitude becomes about 10% of the wavelength  $\lambda = 2\pi/k$ . At this point, higher harmonics of the original sinusoid appear. The perturbed interface is then no longer sinusoidal, but assumes a “bubble-and-spoke” configuration, in which rising, broader bubbles alternate with falling, narrower spikes. The relative width of bubbles and spikes depends on the density ratio of the two fluids, or, equivalently, on the Atwood number  $A$ . When  $A \simeq 1$ , the bubbles are much broader than the spikes. But when  $A \simeq 0$ , that is, the fluids have nearly the same density, there is little distinction between the behavior of bubbles and spikes, and they have nearly the same width.

Eventually the flow reaches a regime which is nearly steady-state, if the initial perturbation was a pure sinusoid. The bubbles rise at constant velocity. If  $A \simeq 1$ , we can carry out an approximate analysis of the resulting flow pattern (following Davies and Taylor 1950, incorporating a suggestion by Layzer 1955) and determine the velocity of the tip of the bubble. Layzer considers the entire history of the instability, from the initial linear stage to the asymptotic steady state, but we focus only on the latter here.

To do so, we employ the concept of *potential flow*. The law of *conservation of circulation* implies that for isentropic flows (that is, flows which are not dissipating or exchanging energy or subjected to shock waves), the curl of the velocity field,  $\nabla \times \mathbf{v}$  (called the *vorticity*) is constant along particle trajectories. In particular, if the vorticity vanishes anywhere on a fluid trajectory, it vanishes everywhere on the trajectory. In the case of an array of bubbles rising into initially motionless fluid, the vorticity of the fluid at a large distance above the bubbles is zero because the fluid is at rest. Even after the fluid begins to fall past the bubbles, its vorticity remains zero, by the law of conservation of circulation. Like any vector field whose curl is zero, the velocity can be therefore be expressed as the gradient of some scalar, by virtue of the vector identity  $\nabla \times (\nabla \phi) = 0$ . This scalar is called the *velocity potential*, and we write  $\mathbf{v} = \nabla \phi$ . This kind of flow is termed potential flow, or *irrotational flow*.

If furthermore we assume that the flow is incompressible, as we did in the linear analysis of Rayleigh-Taylor instability, we have that the velocity is divergenceless:  $\nabla \cdot \mathbf{v} = 0$ . (This follows from the vanishing of the Lagrangian total derivative, Eq. (11), and the continuity equation, Eq. (2).) Therefore, expressing the velocity as the gradient of the potential, we conclude that, for incompressible potential flow, the velocity potential satisfies Laplace's equation:

$$\nabla^2 \phi = 0.$$

Determining the flow field for an array of rising bubbles then amounts to solving Laplace's equation subject to the appropriate boundary conditions.

Another useful relationship for problems of this type is given by *Bernoulli's equation*. It states that, for steady flow of an incompressible fluid,

$$\frac{1}{2} v^2 + \frac{p}{\rho} + gz = \text{constant}$$

along particle trajectories. In our problem, in which  $A \approx 1$ , it is a reasonable approximation to take  $p = \text{constant}$  within the low-density bubble near its tip. Since the high-density fluid at the bubble surface must be in pressure equilibrium with the bubble, and since density is constant in incompressible flow, we can assume that along the surface of the bubble

$$\frac{1}{2} v^2 + gz = \text{constant}. \quad (C1)$$

Let us consider an exactly sinusoidal initial perturbation at an interface, with arbitrary values of the wavevector components  $k_x$  and  $k_y$ . By appropriately rotating the coordinate system in the  $(\hat{x}, \hat{y})$  plane we can make the  $\hat{x}$  direction coincide with the direction of the wavevector  $\mathbf{k}$ , so that  $k_y = 0$  and  $k_x = k$ . Thus the sinusoid varies only in  $\hat{x}$ , and we can ignore the  $\hat{y}$  direction in the following analysis.

From this sort of initial condition, a flow field will eventually arise consisting of an array of identical rising bubbles (which are two-dimensional, like long tunnels, having no variation in  $\hat{y}$ ) arranged with a spatial period of  $\lambda = 2\pi/k$ . The flow pattern is the same as that for a single bubble rising between two parallel frictionless walls located at  $x = \pm \frac{\lambda}{2}$ . The boundary condition at the walls is that the component of the flow velocity normal to the walls vanish there:

$$v_z(x = \pm \frac{\lambda}{2}) = 0.$$

We now transform to the frame-of-reference rising at the same speed as the bubble. Call this speed  $U$ ; the point of this analysis is to determine the value of  $U$ . An additional boundary condition is that in the frame of the bubble, the undisturbed fluid far above the bubble is traveling downward at velocity

$$v_z(z = +\infty) = -U.$$

Solutions of Laplace's equation are well-known from many branches of physics. For geometries such as in our bubble problem, where the flow is two-dimensional and confined

by planar walls, it is clear that a potential of the form

$$\phi(x, z) = -zU - \sum_{n=1}^{\infty} \frac{\lambda a_n}{2n\pi} e^{-\frac{2n\pi z}{\lambda}} \cos\left(\frac{2n\pi x}{\lambda}\right)$$

satisfies Laplace's equation and the boundary conditions just defined. For

$$\frac{\partial \phi}{\partial x} = \sum_{n=1}^{\infty} a_n e^{-\frac{2n\pi z}{\lambda}} \sin\left(\frac{2n\pi x}{\lambda}\right);$$

$$\frac{\partial \phi}{\partial z} = -U + \sum_{n=1}^{\infty} a_n e^{-\frac{2n\pi z}{\lambda}} \cos\left(\frac{2n\pi x}{\lambda}\right);$$

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_{n=1}^{\infty} \frac{2n\pi}{\lambda} a_n e^{-\frac{2n\pi z}{\lambda}} \cos\left(\frac{2n\pi x}{\lambda}\right);$$

$$\frac{\partial^2 \phi}{\partial z^2} = - \sum_{n=1}^{\infty} \frac{2n\pi}{\lambda} a_n e^{-\frac{2n\pi z}{\lambda}} \cos\left(\frac{2n\pi x}{\lambda}\right).$$

Thus

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

as Laplace's equation requires. (Recall that for our choice of coordinate axes,  $\partial \phi / \partial y = \partial^2 \phi / \partial y^2 \equiv 0$ .) Furthermore  $v_x = \partial \phi / \partial x = 0$  at  $x = \pm \frac{\lambda}{2}$  and  $v_z = \partial \phi / \partial z \rightarrow -U$  as  $z \rightarrow \infty$ , as our boundary conditions require.

The trajectory of any fluid particle in the flow field is described by the *stream function*  $\psi$ , which is related to the velocity potential by

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial z},$$

$$\frac{\partial \psi}{\partial z} = -\frac{\partial \phi}{\partial x}.$$

It is easy to verify that the function

$$\psi(x, z) = -xU + \sum_{n=1}^{\infty} \frac{\lambda a_n}{2n\pi} e^{-\frac{2n\pi z}{\lambda}} \sin\left(\frac{2n\pi x}{\lambda}\right)$$

satisfies these relationships. The stream function is constant along particle trajectories for steady flow, so that trajectories are given implicitly by

$$\psi(x, z) = \psi_c = \text{constant}.$$

At  $z = +\infty$ ,  $\psi(x, \infty) = -xU$ , so we see that  $\psi$  is related to the distance of the trajectory from the symmetry plane  $x = 0$  at large distance above the bubble. Thus the trajectory

for a fluid particle that flows down the plane  $x = 0$  and then along the boundary of the bubble is given by  $\psi = 0$ , which implies that

$$\frac{1}{xU} \sum_{n=1}^{\infty} \frac{\lambda a_n}{2n\pi} e^{-\frac{2n\pi z}{\lambda}} \sin\left(\frac{2n\pi x}{\lambda}\right) = 1$$

is the equation of the bubble surface.

A simple approximation to the solution of this problem is obtained, following Davies and Taylor (1950), by keeping only the first term in the sum defining  $\phi$  and  $\psi$ . That is, using  $k = 2\pi/\lambda$ ,

$$\phi = -zU - \frac{a_1}{k} e^{-kx} \cos kx,$$

$$\psi = -xU + \frac{a_1}{k} e^{-kx} \sin kx.$$

Then the velocity components are

$$v_z = \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial z} = a_1 e^{-kx} \sin kx,$$

$$v_x = \frac{\partial \phi}{\partial z} = \frac{\partial \psi}{\partial x} = -U + a_1 e^{-kx} \cos kx.$$

The surface of the bubble is given by

$$e^{kx} = \frac{a_1 \sin kx}{U kx}.$$

The height of the apex of the bubble, at  $x = 0$ , is determined by the values of  $a_1$  and  $U$ . If we demand that the apex occur at  $z = 0$ , then we must have  $a_1 = U$  and so the bubble surface is given by

$$e^{-kx} = \frac{kx}{\sin kx}. \quad (C2)$$

and

$$z = \frac{1}{k} \ln\left(\frac{\sin kx}{kx}\right).$$

To proceed with the solution, we now require that Bernoulli's equation be satisfied along the bubble surface. Inserting the above expressions for the velocity components into Eq. (C1), with  $a_1 = U$ , leads to

$$U^2(e^{-2kx} \sin^2 kx + 1 - 2e^{-kx} \cos kx + e^{-2kx} \cos^2 kx) + 2gz = \text{constant},$$

or

$$e^{-2kx} + 1 - 2e^{-kx} \cos kx + \frac{2gz}{U^2} = 0. \quad (C3)$$

We choose the constant to be zero on the right-hand side of the equation because the apex of the bubble is a stagnation point, with  $v_x = v_y = 0$ , and its height is  $z = 0$ . Thus it is

clear from Eq. (C1) that the constant is zero for this trajectory. Along the bubble surface, Eq. (C2) applies, so we insert that condition into Eq. (C3). The result is

$$\frac{(kx)^2}{\sin^2 kx} - 2 \frac{kx}{\tan kx} + 1 + \frac{2g}{kU^2} \ln\left(\frac{\sin kx}{kx}\right) = 0$$

or

$$u^2 \tan u - 2u \sin^2 u + \sin^2 u \tan u + \frac{2g}{kU^2} \sin^2 u \tan u \ln\left(\frac{\sin u}{u}\right) = 0, \quad (C4)$$

where  $u \equiv kx$ .

Now, for any particular choice of  $g$  and  $U$ , this expression can only be satisfied exactly at a single value of  $x$  in addition to  $x = 0$ . It cannot be satisfied over the entire bubble surface. This is the result of having chosen the simplified potential and stream functions with only the first term of the sum. Keeping more terms in the sum would allow a more complete solution. Nevertheless we can determine a reasonably accurate value for  $U$  by requiring that Eq. (C4) be satisfied in a first-order neighborhood of  $x = 0$ . Accordingly we expand the functions in Eq. (C4):

$$\begin{aligned} \sin^2 u &= u^2 - \frac{u^4}{3} + \mathcal{O}[u^6]; \\ \tan u &= u + \frac{u^3}{3} + \mathcal{O}[u^5]; \\ \sin^2 u \tan u &= u^3 + \mathcal{O}[u^7]; \\ \ln\left(\frac{\sin u}{u}\right) &= -\frac{u^2}{6} - \frac{u^4}{180} + \mathcal{O}[u^6]; \\ \sin^2 u \tan u \ln\left(\frac{\sin u}{u}\right) &= -\frac{u^5}{6} + \mathcal{O}[u^7]. \end{aligned}$$

So Eq. (C4) becomes

$$u^3 + \frac{u^5}{3} - 2u^3 + \frac{2u^5}{3} + u^3 - \frac{2g}{kU^2} \frac{u^5}{6} + \mathcal{O}[u^7] = 0$$

implying that

$$u^5 \left( \frac{1}{3} + \frac{2}{3} - \frac{g}{3kU^2} \right) = 0$$

or

$$U = \sqrt{\frac{g}{3k}} = \sqrt{\frac{g\lambda}{6\pi}} \simeq 0.2303\sqrt{g\lambda}.$$

This is exactly the result of Layzer (1955) for the case of asymptotic steady-state two-dimensional flow between parallel walls. He takes as the length scale the half-distance between the walls  $a = \lambda/2$ , so that

$$U = \sqrt{\frac{ga}{3\pi}} \simeq 0.3257\sqrt{ga}.$$

Layzer also considers the flow of a bubble of circular cross-section, with radius  $R$ . He obtains in this case

$$U \simeq 0.5108\sqrt{gR}.$$

Thus we see that larger bubbles rise faster than smaller bubbles. This dependence is opposite to that for the linear stage of the instability, in which we found that smaller wavelengths grow faster than larger wavelengths.

#### D. Saturation and multimode interactions in intermediate-stage Rayleigh-Taylor instability

The linear analysis of Sec. A depends on the validity of the small-amplitude assumption, that is, on the extent to which first-order quantities are in fact much smaller than the corresponding zeroth-order quantities. However, if the exponential growth that characterizes the linear stage were to persist long enough, the small-amplitude assumption would eventually be violated for any initial perturbation, however small. The departure of the instability evolution from linearity is called *saturation*. We can estimate the conditions required for linearity by considering, for example, the first-order acceleration of a sinusoidal perturbation mode and its relation to the zeroth-order acceleration of gravity  $g$ . The  $z$ -velocity of a pure mode with wavevector  $\mathbf{k}$  oriented along  $\hat{\mathbf{x}}$  can be described by

$$v_z(x, z, t) = W(z)e^{\gamma t} \cos kx,$$

so that the displacement of the interface (initially at  $z = 0$ ) is

$$\eta(x, t) = \int_0^t v_z(x, 0, t') dt' = \frac{1}{\gamma} v_z(x, 0, t).$$

The acceleration of the interface is

$$\frac{\partial v_z(x, 0, t)}{\partial t} = \gamma v_z(x, 0, t) = \gamma^2 \eta(x, t).$$

Linearity requires that this acceleration be much smaller than gravity:  $\gamma^2 \eta \ll g$ . Since the linear growth rate  $\gamma = \sqrt{kgA}$ , this is  $Ak\eta \ll 1$ . Since  $A \leq 1$ , a sufficient condition for linearity is simply

$$k\eta \ll 1.$$

In terms of the wavelength of the mode, this condition is

$$\eta \ll \lambda/2\pi \simeq 0.16\lambda.$$

The consequence of saturation is that the growth of the instability is no longer exponential, but begins to approach the constant-velocity bubble rise typical of late-stage growth. A more stringent estimate of the requirement for linearity comes from estimating the interface displacement at which the linear-stage interface velocity equals the late-stage bubble velocity. As we have just seen, the interface velocity is  $v_z(x, 0, t) = \gamma\eta$ , while the

bubble velocity is  $\sqrt{g/3k}$ . Equating these, for  $A = 1$ , gives  $k\eta = 1/\sqrt{3} \approx 0.58$ , so that linearity requires

$$\eta \ll \frac{1}{\sqrt{3}} \frac{\lambda}{2\pi} \approx 0.09\lambda. \quad (D1)$$

Another consequence of the onset of nonlinearity is that separate perturbation modes on the interface, which grow as if they are isolated during the linear stage, begin to notice one another's presence. This occurs because they begin to affect the zeroth-order flow field which drives the instability; for example, a short-wavelength mode riding along on the bubble of a long-wavelength mode experiences a different effective gravity than the initial  $g$ , because of the additional acceleration in the long-wavelength bubble. This interaction is called *mode coupling*.

Real surfaces in actual ICF experiments have structure at many scales, from millimeters to angstroms. The structure arises for a variety of reasons, including the inherent heterogeneous crystalline structure of materials, as well as marks left by fabrication and machining. When Fourier analyzed, the surfaces typically have a full spectrum, with spectral power at all modes up to some very high wavenumber. An important question arises concerning how saturation occurs in a full spectrum, as opposed to the case of a pure mode just discussed. This is because a group of modes with nearly equal wave vectors can combine constructively over a region of the surface, producing a structure whose net amplitude is much larger than the modes' individual amplitudes. It seems clear that the saturation of this structure should occur when its net amplitude is about 10% of its effective wavelength, as discussed above for pure sinusoids. This means that the individual modes summing to produce this structure must saturate a good deal earlier than we would expect if they were isolated from other modes and individually obeying the inequality (D1). A prescription for determining when modes saturate in a full spectrum was developed by Haan (1989) and is known as the *Haan saturation model*. It expresses a type of modal interaction which is a short-range interaction in wavevector space, involving as it does only neighboring modes which stay in phase over a large enough region to form a structure of significantly higher amplitude than any of the individual modes.

The basic conceptual point of the Haan model is that a pure mode cannot be distinguished from a superposition of several modes except by measurements over a sufficiently large spatial region. The region must be large enough that the individual modes in the superposition have gone out of phase. In regions smaller than this, the saturation of the multimode superposition must occur in the same way as the saturation of the pure mode. For example, consider two modes of nearly equal wavelength ( $\lambda$  and  $\lambda(1 + \epsilon)$ , for example), equal amplitudes, and parallel wavevectors. The modes stay in phase for a large distance because their wavelengths are so nearly equal, but eventually, over a distance  $\lambda/2\epsilon$ , the modes become exactly out of phase. Where they are in phase, they combine to create a net perturbation whose amplitude is twice the individual amplitudes and whose wavelength is approximately  $\lambda$ . When the net perturbation saturates, the two superposed modes clearly have amplitudes which are about half the value of the single-mode saturation amplitude, yet they must individually saturate.

### References

Chandrasekhar

Davies R M and Taylor G, 1950, The mechanics of large bubbles rising through extended liquids and through liquids in tubes, *proc R Soc London A200* 375.

Gamaly

Landau and Lifshitz

Layzer

Takabe H, Mima K, Montierth L, and Morse R L, 1985, Self-consistent growth rate of the Rayleigh-Taylor instability in an ablatively accelerating plasma, *Phys Fluids* 28 3676.