LA-3437

C. 3

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

Spherical Harmonic Reduction of the Fokker-Planck Equation

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

Printed in USA. Price $2.00. Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, United States Department of Commerce, Springfield, Virginia

LA-3437
UC-20, CONTROLLED
THERMONUCLEAR PROCESSES
TID-4500

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

Report written: August 1965

Report distributed: March 1, 1966

Spherical Harmonic Reduction of the Fokker-Planck Equation

by

Thomas A. Oliphant, Jr.

ABSTRACT

The Fokker-Planck equation is reduced to a form that is useful from the viewpoint of doing practical calculations of problems involving configuration space as well as velocity space. The basic technique is a spherical harmonic decomposition in velocity space that reduces the number of independent variables by two. As an example, we show how to apply this method to a problem with theta-pinch geometry.

ACKNOWLEDGEMENTS

The author is indebted to F. L. Ribe and D. A. Baker of this laboratory for their continued interest and encouragement.

TABLE OF CONTENTS

I. INTRODUCTION

Realistic calculations of the plasma behavior in the theta-pinch and other devices have been undertaken in the magnetohydrodynamics approximation.[1,2,3] In comparison studies with the Scylla-IV theta-pinch,[3] we find that the dynamics of the plasma occur so rapidly that the assumption of ion-ion collision equilibrium cannot be expected to be valid throughout much of the calculation where plasma temperatures are high, although it is valid at the beginning stages. In fact, for very high temperatures at peak compression of the plasma, we suspect that the plasma may even act somewhat like a collisionless plasma. Therefore, ideally, we should study the Boltzmann equation instead of the MHD equations, so that we can operate in the regions where the collision term is relatively unimportant, in the region where the collision term dominates, and in the transition regions between the two. Because of the relative unimportance of large-angle collisions in the fully ionized plasma, we feel justified in using the Fokker-Planck approximation to the collision term.

The purpose of this paper is to rigorously reduce the Fokker-Planck equation to numerically tractable form. The procedure that allows us to do this is the spherical harmonic expansion in velocity space. Rosenbluth, MacDonald, and Judd have shown how to write the Fokker-Planck collision term in spherical coordinates in velocity space.[4] They assume symmetry about the spherical harmonic axis, so that their azimuthal angle drops out of consideration. We cannot assume such symmetry; however, the expression obtained below with the azimuthal dependence included is not much more complicated. They then expand their collision term in Legendre polynomials. Because of our lower symmetry, we must use associated Legendre polynomials. Krook has used the spherical harmonic expansion in connection with a method involving velocity moments.[5] Our method, in addition to being numerically tractable, is rigorous in that we include as many spherical harmonic terms as are necessary. We anticipate that there will be many situations in which the number of spherical harmonics required will be quite reasonable.

In practical calculations we will, among other things, be interested in the development and propagation of shock waves as they occur in plasma dynamics, for example, in the theta-pinch problem. The transport theory of shock waves is far from having been completely exploited, although much work has been done.[5-10] The procedure outlined in this paper gives the means to apply the Fokker-Planck approach to the nonequilibrium aspects of shock waves and related dynamical phenomena in plasmas for various configuration space geometries.

II. THE BOLTZMANN TRANSPORT EQUATION

Let us consider a gas composed of n components. The Boltzmann transport equation for this problem can be written in the following way:

$$\mathcal{D} f_a = \sum_{b=1}^{n} \left. \frac{\partial f_a}{\partial t} \right|_{ab} \quad (a = 1, \ldots, n) \tag{2.1}$$

where the symbol on the left-hand side stands for the phase-space velocity derivative or, in other words, the transport term; and the symbol on the right-hand side stands for the collision term. In this section we will discuss the transport term, and in Section III we will discuss the collision term. Because the transport term involves only one subscript, a, we will omit it in the remainder of this section. Therefore, we write

$$\mathcal{D} f = \frac{\partial f}{\partial t} + \underset{\sim}{V} \cdot \nabla f + \underset{\sim}{a} \cdot \nabla_V f \tag{2.2}$$

where $\underset{\sim}{V}$ is the velocity, and $\underset{\sim}{a}$ is the acceleration of an individual particle. We assume that the acceleration $\underset{\sim}{a}$ is that of a charged particle moving in an external electromagnetic field:

$$\underset{\sim}{a} = \frac{q}{m} \left[\underset{\sim}{E}(\underset{\sim}{x}) + \frac{\underset{\sim}{V} \times \underset{\sim}{B}(\underset{\sim}{x})}{c} \right] \tag{2.3}$$

The nature and origin of the electromagnetic field appearing in (2.3) will be discussed in more detail in Section IV. The average velocity $\overline{\underset{\sim}{V}}$ is given by

$$\overline{\underset{\sim}{V}}(\underset{\sim}{x},t) = \int \underset{\sim}{V}\, f\, (\underset{\sim}{x},\underset{\sim}{V},t)\, d\underset{\sim}{V} \tag{2.4}$$

We can define the special velocity $\underset{\sim}{v}$ by

$$\underset{\sim}{v}(x,t) = \underset{\sim}{V} - \overline{\underset{\sim}{V}}(\underset{\sim}{x},t) \tag{2.5}$$

However, we prefer to introduce an extra generality into the theory by defining $\underset{\sim}{v}(\underset{\sim}{x},t)$ in terms of an arbitrary velocity function $\underset{\sim}{u}(\underset{\sim}{x},t)$:

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{V} - \underset{\sim}{u}(\underset{\sim}{x},t) \tag{2.6}$$

Thus, we can go through the formalism just as easily and later, if we wish, impose the condition

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \overline{\underset{\sim}{V}}(\underset{\sim}{x},t) \tag{2.7}$$

Alternatively, we keep in mind the possibility that we may find it convenient to interpret $\underset{\sim}{u}(\underset{\sim}{x},t)$ in some other way.

Our purpose is now to regard f as a function of $(\underset{\sim}{x},\underset{\sim}{v},t)$. However, we must keep in mind that $\underset{\sim}{v}$ is a function of $(\underset{\sim}{x},t)$ and write

$$f = f\left(\underset{\sim}{x},\underset{\sim}{v}(\underset{\sim}{x},t),t\right) \tag{2.8}$$

Therefore, in (2.2) we must make the following substitutions (Ref. 11, §3.13):

$$\frac{\partial f}{\partial t} \rightarrow \frac{\partial f}{\partial t} - (\nabla_v f) \cdot \frac{\partial \underset{\sim}{u}}{\partial t} \tag{2.9}$$

$$\underset{\sim}{V} \cdot \nabla f \rightarrow \underset{\sim}{V} \cdot \nabla f - (\nabla_V f) \cdot [(\underset{\sim}{V} \cdot \nabla)\underset{\sim}{u}] \tag{2.10}$$

$$\underset{\sim}{a} \cdot \nabla_V f \rightarrow \underset{\sim}{a} \cdot \nabla_V f \tag{2.11}$$

Thus, (2.2) can be written in the form

$$\mathcal{D}f = \frac{Df}{Dt} + \underset{\sim}{v} \cdot \nabla f + (\nabla_V f) \cdot \left(\underset{\sim}{a} - \frac{D\underset{\sim}{u}}{Dt}\right) - (\nabla_V f) \cdot [(\underset{\sim}{v} \cdot \nabla)\underset{\sim}{u}] \tag{2.12}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla \tag{2.13}$$

(compare Ref. 11, page 50).

We will restrict our configuration space coordinates to be an orthogonal, curvilinear system denoted by

$$\underset{\sim}{x} = (\eta_1, \eta_2, \eta_3) \tag{2.14}$$

All vectors can be expressed in terms of their projections along the local, orthogonal basis vectors $\hat{\underset{\sim}{\varepsilon}}_i$ of the systems (η_1, η_2, η_3). In this system we write the gradient operator as [12]

$$\nabla = \frac{\hat{\underset{\sim}{\varepsilon}}_i}{\eta_i} \frac{\partial}{\partial \eta_i} \tag{2.15}$$

We observe the summation convention for repeated indices referring to coordinate components. Here

$$h_i = h_i(\eta_j) \tag{2.16}$$

In terms of these coordinates, (2.12) and (2.13) are written explicitly as follows:

$$\mathscr{D}f = \frac{Df}{Dt} + \frac{v_i}{h_i}\frac{\partial f}{\partial \eta_i} + (\nabla_v f)\cdot\left(\underset{\sim}{a} - \frac{D\underset{\sim}{u}}{Dt} - \frac{v_i}{h_i}\frac{\partial \underset{\sim}{u}}{\partial \eta_i}\right) \tag{2.17}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u_i}{h_i}\frac{\partial}{\partial \eta_i} \tag{2.18}$$

Using the local, orthogonal basis of the curvilinear coordinates as a basis for a system of rectangular velocity coordinates (v_1, v_2, v_3), we proceed to transform to a system of orthogonal, curvilinear velocity coordinates (ξ_1, ξ_2, ξ_3). Thus, we will have

$$f = f(\eta_i, \xi_i, t) \tag{2.19}$$

and

$$\nabla_v = \frac{\hat{\underset{\sim}{e}}_i}{g_i}\frac{\partial}{\partial \xi_i} \tag{2.20}$$

where

$$g_i = g_i(\xi_j) \tag{2.21}$$

We must be careful to note that the basic vectors $\hat{\underset{\sim}{e}}_i$ are different from the basic vectors $\hat{\underset{\sim}{\varepsilon}}_i$ of (2.15). We may now write (2.17) in the form

$$\mathscr{D}f = \frac{Df}{Dt} + \frac{v_i}{h_i}\frac{\partial f}{\partial \eta_i} + \frac{1}{g_i}\left(a_j - \frac{Du_j}{Dt} - \frac{v_k}{h_k}\frac{\partial u_j}{\partial \eta_k}\right)\frac{\partial \xi_i}{\partial v_j}\frac{\partial f}{\partial \xi_i} \tag{2.22}$$

For our system ξ_i we choose, in particular, the spherical coordinate system. Thus, we may write

$$\left.\begin{aligned} v_1 &= v \sin\vartheta \cos\varphi \\ v_2 &= v \sin\vartheta \sin\varphi \\ v_3 &= v \cos\vartheta \end{aligned}\right\} \tag{2.23}$$

We use the script characters ϑ and φ for the angles in velocity space to distinguish them from the angles θ and Φ in configuration space. Actually, we find it more convenient to work with the cos w equal to cos ϑ instead of to ϑ. Thus, we define our system ξ_i by

$$\left.\begin{aligned} \xi_1 &= v \\ \xi_2 &= w = \cos\vartheta \\ \xi_3 &= \varphi \end{aligned}\right\} \qquad (2.24)$$

and write the transformation in the form

$$\left.\begin{aligned} v_1 &= v\sqrt{1 - w^2}\cos\varphi \\ v_2 &= v\sqrt{1 - w^2}\sin\varphi \\ v_3 &= vw \end{aligned}\right\} \qquad (2.25)$$

For convenience we will sometimes use the ξ_i notation, although we actually refer to the specific system (v,w,φ) as implied by (2.24).

We now write out (2.22) explicitly in terms of our spherical velocity coordinates. In addition to (2.3), (2.6), and (2.25) we will need the following relations:

$$\left.\begin{aligned} g_v &= 1 \\ g_w &= \frac{1}{\sqrt{1 - w^2}} \\ g_\varphi &= v\sqrt{1 - w^2} \end{aligned}\right\} \tag{2.26}$$

$$\left.\begin{aligned} &\frac{\partial v}{\partial v_1} = \sqrt{1 - w^2}\cos\varphi \qquad \frac{\partial w}{\partial v_1} = -\frac{w\sqrt{1 - w^2}}{v}\cos\varphi \qquad \frac{\partial\varphi}{\partial v_1} = -\frac{\sin\varphi}{v\sqrt{1 - w^2}} \\ &\frac{\partial v}{\partial v_2} = \sqrt{1 - w^2}\sin\varphi \qquad \frac{\partial w}{\partial v_2} = -\frac{w\sqrt{1 - w^2}}{v}\sin\varphi \qquad \frac{\partial\varphi}{\partial v_2} = \frac{\cos\varphi}{v\sqrt{1 - w^2}} \\ &\frac{\partial v}{\partial v_3} = w \qquad \frac{\partial w}{\partial v_3} = \frac{1 - w^2}{v} \qquad \frac{\partial\varphi}{\partial v_3} = 0 \end{aligned}\right\} \tag{2.27}$$

Substituting (2.3), (2.6) and (2.25) - (2.26) - (2.27) into (2.22), we get

$$\mathcal{D}f = \frac{Df}{Dt} + \frac{v\sqrt{1 - w^2}\cos\varphi}{h_1}\frac{\partial f}{\partial\eta_1} + \frac{v\sqrt{1 - w^2}\sin\varphi}{h_2}\frac{\partial f}{\partial\eta_2} + \frac{vw}{h_3}\frac{\partial f}{\partial\eta_3}$$

$$+ \left\{\left(\frac{qE_1^*}{m} - \frac{Du_1}{Dt}\right)\sqrt{1 - w^2}\cos\varphi + \left(\frac{qE_2^*}{m} - \frac{Du_2}{Dt}\right)\sqrt{1-w^2}\sin\varphi + \left(\frac{qE_3^*}{m} - \frac{Du_3}{Dt}\right)w\right.$$

$$- \frac{1}{h_1}\frac{\partial u_1}{\partial\eta_1}v(1 - w^2)\cos^2\varphi - \frac{1}{h_2}\frac{\partial u_1}{\partial\eta_2}v(1 - w^2)\sin\varphi\cos\varphi$$

$$- \frac{1}{h_3} \frac{\partial u_1}{\partial \eta_3} vw\sqrt{1-w^2} \cos\varphi - \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_1} v(1-w^2) \sin\varphi \cos\varphi$$

$$- \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_2} v(1-w^2) \sin^2\varphi - \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} vw\sqrt{1-w^2} \sin\varphi$$

$$- \frac{1}{h_1} \frac{\partial u_3}{\partial \eta_1} vw\sqrt{1-w^2} \cos\varphi - \frac{1}{h_2} \frac{\partial u_3}{\partial \eta_2} vw\sqrt{1-w^2} \sin\varphi - \frac{1}{h_3} \frac{\partial u_3}{\partial \eta_3} vw^2 \Big\} \frac{\partial f}{\partial v}$$

$$+ \Big\{ - \left(\frac{qE_1^*}{m} - \frac{Du_1}{Dt} \right) \frac{w(1-w^2)}{v^2} \cos\varphi - \left(\frac{qE_2^*}{m} - \frac{Du_2}{Dt} \right) \frac{w(1-w^2)}{v^2} \sin\varphi$$

$$+ \left(\frac{qE_3^*}{m} - \frac{Du_3}{Dt} \right) \frac{(1-w^2)^{3/2}}{v^2} + \frac{1}{h_1} \frac{\partial u_1}{\partial \eta_1} \frac{w(1-w^2)^{3/2}}{v} \cos^2\varphi$$

$$+ \frac{1}{h_2} \frac{\partial u_1}{\partial \eta_2} \frac{w(1-w^2)^{3/2}}{v} \sin\varphi \cos\varphi + \frac{1}{h_3} \frac{\partial u_1}{\partial \eta_3} \frac{w^2(1-w^2)}{v} \cos\varphi$$

$$+ \frac{1}{h_1} \frac{\partial u_2}{\partial \eta_1} \frac{w(1-w^2)^{3/2}}{v} \sin\varphi \cos\varphi + \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_2} \frac{w(1-w^2)^{3/2}}{v} \sin^2\varphi$$

$$+ \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} \frac{w^2(1-w^2)}{v} \sin\varphi - \frac{1}{h_1} \frac{\partial u_3}{\partial \eta_1} \frac{(1-w^2)^2}{v} \cos\varphi$$

$$- \frac{1}{h_2} \frac{\partial u_3}{\partial \eta_2} \frac{(1-w^2)^2}{v} \sin\varphi - \frac{1}{h_3} \frac{\partial u_3}{\partial \eta_3} \frac{w(1-w^2)^{3/2}}{v} - \frac{qB_1}{mc} \frac{(1-w^2)}{v} \sin\varphi$$

$$+ \frac{qB_2}{mc} \frac{(1-w^2)}{v} \cos\varphi \Big\} \frac{\partial f}{\partial w} + \Big\{ - \left(\frac{qE_1^*}{m} - \frac{Du_1}{Dt} \right) \frac{\sin\varphi}{v^2(1-w^2)}$$

$$+ \left(\frac{qE_2^*}{m} - \frac{Du_2}{Dt}\right) \frac{\cos \varphi}{v^2(1-w^2)} + \frac{1}{h_1} \frac{\partial u_1}{\partial \eta_1} \frac{\sin \varphi \cos \varphi}{v\sqrt{1-w^2}} + \frac{1}{h_2} \frac{\partial u_1}{\partial \eta_2} \frac{\sin^2 \varphi}{v\sqrt{1-w^2}}$$

$$+ \frac{1}{h_3} \frac{\partial u_1}{\partial \eta_3} \frac{w}{v(1-w^2)} \sin \varphi - \frac{1}{h_1} \frac{\partial u_2}{\partial \eta_1} \frac{\cos^2 \varphi}{v\sqrt{1-w^2}} - \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_2} \frac{\sin \varphi \cos \varphi}{v\sqrt{1-w^2}}$$

$$- \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} \frac{w}{v(1-w^2)} \cos \varphi + \frac{qB_1}{mc} \frac{w}{v(1-w^2)} \cos \varphi + \frac{qB_2}{mc} \frac{w}{v(1-w^2)} \sin \varphi$$

$$\left. - \frac{qB_3}{mc} \frac{1}{v\sqrt{1-w^2}} \right\} \frac{\partial f}{\partial \varphi} \tag{2.28}$$

where

$$\underset{\sim}{E}^* = \underset{\sim}{E} + \frac{\underset{\sim}{u} \times \underset{\sim}{B}}{c} \tag{2.29}$$

We now introduce a spherical harmonic expansion in velocity space:

$$f(\eta_j, \xi_j, t) = \sum_{\ell m} f_{\ell m}(\eta_j, v, t) \; Y_{\ell m} \; (w, \varphi) \tag{2.30}$$

Because the $Y_{\ell m}$ are complex, we will have complex components $f_{\ell m}$ even though the complete function f is real. In order to write our result completely in terms of real quantities, we introduce

$$f_{\ell m} = f^+_{\ell m} + i \; f^-_{\ell m} \tag{2.31}$$

where $f^{\pm}_{\ell m}$ are both real. Substituting (2.30) and (2.31) into (2.28), multiplying on the left by $Y^*_{\ell m}$, and integrating over the solid angle

$d\Omega = dw\, d\varphi$, we obtain

$$(\mathcal{D}f)_{\ell m} = (\mathcal{D}f)^{+}_{\ell m} + i\,(\mathcal{D}f)^{-}_{\ell m} \qquad (2.32)$$

where

$$(\mathcal{D}f)^{\pm}_{\ell m} = \frac{Df^{\pm}_{\ell m}}{Dt} + \sum_{\ell'} \Bigg[\frac{v}{h_1}\left({}_1I^{\ell',m+1}_{\ell m} \frac{\partial f^{\pm}_{\ell',m+1}}{\partial \eta_1} + {}_1I^{\ell',m+1}_{\ell m} \frac{\partial f^{\pm}_{\ell',m-1}}{\partial \eta_1} \right)$$

$$\mp \frac{v}{h_2}\left({}_1I^{\ell',m+1}_{\ell m} \frac{\partial f^{\mp}_{\ell',m+1}}{\partial \eta_2} - {}_1I^{\ell',m-1}_{\ell m} \frac{\partial f^{\mp}_{\ell',m-1}}{\partial \eta_2} \right) + \frac{v}{h_3}\left({}_2I^{\ell' m}_{\ell m} \frac{\partial f^{\pm}_{\ell' m}}{\partial \eta_3} \right)$$

$$+\left(\frac{qE_1}{m} - \frac{Du_1}{Dt} \right)\left({}_1I^{\ell',m+1}_{\ell m} \frac{\partial f^{\pm}_{\ell',m+1}}{\partial v} + {}_1I^{\ell',m-1}_{\ell m} \frac{\partial f^{\pm}_{\ell',m-1}}{\partial v} \right)$$

$$\mp\left(\frac{qE^*_2}{m} - \frac{Du_2}{Dt} \right)\left({}_1I^{\ell',m+1}_{\ell m} \frac{\partial f^{\mp}_{\ell',m+1}}{\partial v} - {}_1I^{\ell',m-1}_{\ell m} \frac{\partial f^{\mp}_{\ell',m-1}}{\partial v} \right)$$

$$+\left(\frac{qE^*_3}{m} - \frac{Du_3}{Dt} \right)\left({}_2I^{\ell' m}_{\ell m} \frac{\partial f^{\pm}_{\ell' m}}{\partial v} \right)$$

$$-\left(\frac{qE^*_1}{m} - \frac{Du_1}{Dt} \right)\frac{1}{v^2}\left({}_3I^{\ell',m+1}_{\ell m}\, f^{\pm}_{\ell',m+1} + {}_3I^{\ell',m-1}_{\ell m}\, f^{\pm}_{\ell',m-1} \right)$$

$$\pm\left(\frac{qE^*_2}{m} - \frac{Du_2}{Dt} \right)\frac{1}{v^2}\left({}_3I^{\ell',m+1}_{\ell m}\, f^{\mp}_{\ell',m+1} - {}_3I^{\ell',m-1}_{\ell m}\, f^{\mp}_{\ell',m-1} \right)$$

$$+\left(\frac{qE^*_3}{m} - \frac{Du_3}{Dt} \right)\frac{1}{v^2}\left({}_4I^{\ell' m}_{\ell m}\, f^{\pm}_{\ell' m} \right)$$

$$+ \left(\frac{qE_1^*}{m} - \frac{Du_1}{Dt}\right)\frac{1}{v^2}\left({}_6I_{\ell m}^{\ell',m+1}\, f^{\pm}_{\ell',m+1} - {}_6I_{\ell m}^{\ell',m-1}\, f^{\pm}_{\ell',m-1}\right)$$

$$\mp \left(\frac{qE_2^*}{m} - \frac{Du_2}{Dt}\right)\frac{1}{v^2}\left({}_6I_{\ell m}^{\ell',m+1}\, f^{\mp}_{\ell',m+1} + {}_6I_{\ell m}^{\ell',m-1}\, f^{\mp}_{\ell',m-1}\right)$$

$$\pm \frac{qB_1}{mcv}\left({}_5I_{\ell m}^{\ell',m+1}\, f^{\pm}_{\ell',m+1} - {}_5I_{\ell m}^{\ell',m-1}\, f^{\mp}_{\ell',m-1}\right)$$

$$+ \frac{qB_2}{mcv}\left({}_5I_{\ell m}^{\ell',m+1}\, f^{\pm}_{\ell',m+1} + {}_5I_{\ell m}^{\ell',m-1}\, f^{\pm}_{\ell',m-1}\right)$$

$$\mp \frac{qB_1}{mcv}\left({}_7I_{\ell m}^{\ell',m+1}\, f^{\mp}_{\ell',m+1} + {}_7I_{\ell m}^{\ell',m-1}\, f^{\mp}_{\ell',m-1}\right)$$

$$- \frac{qB_2}{mcv}\left({}_7I_{\ell m}^{\ell',m+1}\, f^{\pm}_{\ell',m+1} - {}_7I_{\ell m}^{\ell',m-1}\, f^{\pm}_{\ell',m-1}\right)$$

$$\pm \frac{qB_3}{mcv}\left({}_8I_{\ell m}^{\ell' m}\, f^{\mp}_{\ell' m}\right)$$

$$- \frac{1}{h_1}\frac{\partial u_1}{\partial \eta_1} v\left({}_9I_{\ell m}^{\ell',m+2}\frac{\partial f^{\pm}_{\ell',m+2}}{\partial v} + {}_9I_{\ell m}^{\ell',m-2}\frac{\partial f^{\pm}_{\ell',m-2}}{\partial v} + 2\,{}_9I_{\ell m}^{\ell' m}\frac{\partial f^{\pm}_{\ell' m}}{\partial v}\right)$$

$$\pm \frac{1}{h_2}\frac{\partial u_1}{\partial \eta_2} v\left({}_9I_{\ell m}^{\ell',m+2}\frac{\partial f^{\mp}_{\ell',m+2}}{\partial v} - {}_9I_{\ell m}^{\ell',m-2}\frac{\partial f^{\mp}_{\ell',m-2}}{\partial v}\right)$$

$$- \frac{1}{h_3}\frac{\partial u_1}{\partial \eta_3} v\left({}_{10}I_{\ell m}^{\ell',m+1}\frac{\partial f^{\pm}_{\ell',m+1}}{\partial v} + {}_{10}I_{\ell m}^{\ell',m-1}\frac{\partial f^{\pm}_{\ell',m-1}}{\partial v}\right)$$

$$\pm \frac{1}{h_1} \frac{\partial u_2}{\partial \eta_1} v \left({}_9I_{\ell m}^{\ell',m+2} \frac{\partial f^{\pm}_{\ell',m+2}}{\partial v} - {}_9I_{\ell m}^{\ell',m-2} \frac{\partial f^{\pm}_{\ell',m-2}}{\partial v} \right)$$

$$+ \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_2} v \left({}_9I_{\ell m}^{\ell',m+2} \frac{\partial f^{\pm}_{\ell',m+2}}{\partial v} + {}_9I_{\ell m}^{\ell',m-2} \frac{\partial f^{\pm}_{\ell',m-2}}{\partial v} - 2\ {}_9I_{\ell m}^{\ell' m} \frac{\partial f^{\pm}_{\ell' m}}{\partial v} \right)$$

$$\pm \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} v \left({}_{10}I_{\ell m}^{\ell',m+1} \frac{\partial f^{\mp}_{\ell',m+1}}{\partial v} - {}_{10}I_{\ell m}^{\ell',m-1} \frac{\partial f^{\mp}_{\ell',m-1}}{\partial v} \right)$$

$$- \frac{1}{h_1} \frac{\partial u_3}{\partial \eta_1} v \left({}_{10}I_{\ell m}^{\ell',m+1} \frac{\partial f^{\pm}_{\ell',m+1}}{\partial v} + {}_{10}I_{\ell m}^{\ell',m-1} \frac{\partial f^{\pm}_{\ell',m-1}}{\partial v} \right)$$

$$\pm \frac{1}{h_2} \frac{\partial u_3}{\partial \eta_2} v \left({}_{10}I_{\ell m}^{\ell',m+1} \frac{\partial f^{\mp}_{\ell',m+1}}{\partial v} - {}_{10}I_{\ell m}^{\ell',m-1} \frac{\partial f^{\mp}_{\ell',m-1}}{\partial v} \right)$$

$$- \frac{1}{h_3} \frac{\partial u_3}{\partial \eta_3} v \left({}_{11}I_{\ell m}^{\ell' m} \frac{\partial f^{\pm}_{\ell' m}}{\partial v} \right)$$

$$+ \frac{1}{h_1} \frac{\partial u_1}{\partial \eta_1} \frac{1}{v} \left({}_{12}I_{\ell m}^{\ell',m+2} f^{\pm}_{\ell',m+2} + {}_{12}I_{\ell m}^{\ell',m-2} f^{\pm}_{\ell',m-2} + 2\ {}_{12}I_{\ell m}^{\ell' m} f^{\pm}_{\ell' m} \right)$$

$$\mp \frac{1}{h_2} \frac{\partial u_1}{\partial \eta_2} \frac{1}{v} \left({}_{12}I_{\ell m}^{\ell',m+2} f^{\mp}_{\ell',m+2} - {}_{12}I_{\ell m}^{\ell',m-2} f^{\mp}_{\ell',m-2} \right)$$

$$+ \frac{1}{h_3} \frac{\partial u_1}{\partial \eta_3} \frac{1}{v} \left({}_{13}I_{\ell m}^{\ell',m+1} f^{\pm}_{\ell',m+1} + {}_{13}I_{\ell m}^{\ell',m-1} f^{\pm}_{\ell',m-1} \right)$$

$$\mp \frac{1}{h_1} \frac{\partial u_2}{\partial \eta_1} \frac{1}{v} \left({}_{12}I_{\ell m}^{\ell',m+2} f^{\mp}_{\ell',m+2} - {}_{12}I_{\ell m}^{\ell',m-2} f^{\mp}_{\ell',m-2} \right)$$

$$- \frac{1}{h_2} \frac{\partial u_2}{\partial \eta_2} \frac{1}{v} \left({}_{12}I_{\ell m}^{\ell',m+2} f^{\pm}_{\ell',m+2} + {}_{12}I_{\ell m}^{\ell',m-2} f^{\pm}_{\ell',m-2} - 2\ {}_{12}I_{\ell m}^{\ell' m} f^{\pm}_{\ell' m} \right)$$

$$\mp \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} \frac{1}{v} \left({}_{13}I_{\ell m}^{\ell',m+1} f^{\mp}_{\ell',m+1} - {}_{13}I_{\ell m}^{\ell',m-1} f^{\mp}_{\ell',m-1} \right)$$

$$- \frac{1}{h_1} \frac{\partial u_3}{\partial \eta_1} \frac{1}{v} \left({}_{14}I_{\ell m}^{\ell',m+1} f^{\pm}_{\ell',m+1} + {}_{14}I_{\ell m}^{\ell',m-1} f^{\pm}_{\ell',m-1} \right)$$

$$\pm \frac{1}{h_2} \frac{\partial u_3}{\partial \eta_2} \frac{1}{v} \left({}_{14}I_{\ell m}^{\ell',m+1} f^{\mp}_{\ell',m+1} - {}_{14}I_{\ell m}^{\ell',m-1} f^{\mp}_{\ell',m-1} \right)$$

$$- \frac{4}{h_3} \frac{\partial u_3}{\partial \eta_3} \frac{1}{v} \left({}_{12}I_{\ell m}^{\ell' m} f^{\pm}_{\ell' m} \right)$$

$$- \frac{1}{2h_1} \frac{\partial u_1}{\partial \eta_1} \frac{1}{v} \left({}_{8}I_{\ell m}^{\ell',m+2} f^{\pm}_{\ell',m+2} - {}_{8}I_{\ell m}^{\ell',m-2} f^{\pm}_{\ell',m-2} \right)$$

$$\pm \frac{1}{2h_2} \frac{\partial u_1}{\partial \eta_2} \frac{1}{v} \left({}_{8}I_{\ell m}^{\ell',m+2} f^{\mp}_{\ell',m+2} + {}_{8}I_{\ell m}^{\ell',m-2} f^{\mp}_{\ell',m-2} - 2\ {}_{8}I_{\ell m}^{\ell' m} f^{\mp}_{\ell' m} \right)$$

$$- \frac{1}{h_3} \frac{\partial u_1}{\partial \eta_3} \frac{1}{v} \left({}_{7}I_{\ell m}^{\ell',m+1} f^{\pm}_{\ell',m+1} - {}_{7}I_{\ell m}^{\ell',m-1} f^{\pm}_{\ell',m-1} \right)$$

$$\pm \frac{1}{2h_1} \frac{\partial u_2}{\partial \eta_1} \frac{1}{v} \left({}_{8}I_{\ell m}^{\ell',m+2} f^{\mp}_{\ell',m+2} + {}_{8}I_{\ell m}^{\ell',m-2} f^{\mp}_{\ell',m-2} + 2\ {}_{8}I_{\ell m}^{\ell' m} f^{\mp}_{\ell' m} \right)$$

$$+ \frac{1}{2h_2} \frac{\partial u_2}{\partial \eta_2} \frac{1}{v} \left({}_{8}I_{\ell m}^{\ell',m+2} f^{\pm}_{\ell',m+2} - {}_{8}I_{\ell m}^{\ell',m-2} f^{\pm}_{\ell',m-2} \right)$$

$$\pm \quad \frac{1}{h_3} \frac{\partial u_2}{\partial \eta_3} \frac{1}{v} \left({}_7I_{\ell m}^{\ell',m+1} f_{\ell',m+1}^{\mp} + {}_7I_{\ell m}^{\ell',m-1} f_{\ell',m-1}^{\mp} \right) \Bigg] \qquad (2.33)$$

$$(\mathcal{D}f)_{\ell m}^{\pm} = \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} \int Y_{\ell m}^{*} \, (\mathcal{D}f) \quad d\Omega \qquad (2.34)$$

$$\left.\begin{aligned}
{}_1I_{\ell m}^{\ell'm'} &= \pi a_{\ell m} \, a_{\ell'm'} \int \sqrt{1-w^2} \; P_\ell^m \, P_{\ell'}^{m'} \, dw \\
{}_2I_{\ell m}^{\ell'm'} &= 2\pi a_{\ell m} \, a_{\ell'm'} \int w \; P_\ell^m \, P_{\ell'}^{m'} \, dw \\
{}_3I_{\ell m}^{\ell'm'} &= \pi a_{\ell m} \, a_{\ell'm'} \int w(1-w^2) \; P_\ell^m \, \frac{dP_{\ell'}^{m'}}{dw} \, dw \\
{}_4I_{\ell m}^{\ell'm'} &= 2\pi a_{\ell m} \, a_{\ell'm'} \int (1-w^2)^{3/2} \; P_\ell^m \, \frac{dP_{\ell'}^{m'}}{dw} \, dw \\
{}_5I_{\ell m}^{\ell'm'} &= \pi a_{\ell m} \, a_{\ell'm'} \int (1-w^2) \; P_\ell^m \, \frac{dP_{\ell'}^{m'}}{dw} \, dw \\
{}_6I_{\ell m}^{\ell'm'} &= \pi m' \, a_{\ell m} \, a_{\ell'm'} \int \frac{1}{(1-w^2)} \; P_\ell^m \, P_{\ell'}^{m'} \, dw \\
{}_7I_{\ell m}^{\ell'm'} &= \pi m' a_{\ell m} \, a_{\ell'm'} \int \frac{w}{1-w^2} \; P_\ell^m \, P_{\ell'}^{m'} \, dw \\
{}_8I_{\ell m}^{\ell'm'} &= \pi m' a_{\ell m} \, a_{\ell'm'} \int \frac{1}{\sqrt{1-w^2}} \; P_\ell^m \, P_{\ell'}^{m'} \, dw
\end{aligned}\right\} \qquad (2.35)$$

$$
\left.
\begin{aligned}
{}_{9}I_{\ell m}^{\ell' m'} &= \frac{\pi}{2}\; a_{\ell m}\, a_{\ell' m'} \int (1-w^2)\, P_\ell^m\, P_{\ell'}^{m'}\, dw \\
{}_{10}I_{\ell m}^{\ell' m'} &= \pi a_{\ell m}\, a_{\ell' m'} \int w\sqrt{1-w^2}\, P_\ell^m\, P_{\ell'}^{m'}\, dw \\
{}_{11}I_{\ell m}^{\ell' m'} &= 2\pi a_{\ell m}\, a_{\ell' m'} \int w^2\, P_\ell^m\, P_{\ell'}^{m'}\, dw \\
{}_{12}I_{\ell m}^{\ell' m'} &= \frac{\pi}{2}\, a_{\ell m}\, a_{\ell' m'} \int w(1-w^2)^{3/2}\, P_\ell^m\, \frac{dP_{\ell'}^{m'}}{dw}\, dw \\
{}_{13}I_{\ell m}^{\ell' m'} &= \pi a_{\ell m}\, a_{\ell' m'} \int w^2(1-w^2)\, P_\ell^m\, \frac{dP_{\ell'}^{m'}}{dw}\, dw \\
{}_{14}I_{\ell m}^{\ell' m'} &= \pi a_{\ell m}\, a_{\ell' m'} \int (1-w^2)^2\, P_\ell^m\, \frac{dP_{\ell'}^{m'}}{dw}\, dw
\end{aligned}
\right\}
$$

and

$$
a_{\ell m} \left\{
\begin{array}{ll}
(-)^m \sqrt{\dfrac{2\ell+1}{4\pi}\,\dfrac{(\ell-|m|)!}{(\ell+|m|)!}}\;, & m \geq 0 \\[3ex]
\sqrt{\dfrac{2\ell+1}{4\pi}\,\dfrac{(\ell-|m|)!}{(\ell+|m|)!}} & m \leq 0
\end{array}
\right\} \qquad (2.36)
$$

In (2.36) we have used the Condon and Shortley phase convention.[13] In summary, let us point out what we have accomplished up to this point. For each component of the gas (a=1, •••, n) we have expanded the transport quantity δf_a -defined in (2.2)- in terms of spherical harmonics and taken the projections on a given spherical harmonic. At the expense of introducing the (ℓ,m) indices, we have removed two of the velocity variables. The independent variables of our problem are now $(\underset{\sim}{x},v,t)$. To complete the project we must treat the collision term in the same manner. This is done in the next section.

III. THE FOKKER-PLANCK COLLISION TERM

As mentioned in the introduction, we treat the collision term in the Fokker-Planck approximation and restrict our considerations to the fully ionized plasma. Therefore, we write the collision term in the form[5,14]

$$\frac{1}{A_a} \left.\frac{\partial f_a(\underset{\sim}{x},\underset{\sim}{v},t)}{\partial t}\right|_{ab} = 4\pi\left(\frac{m_a}{m} - 1\right) f_a(\underset{\sim}{x},\underset{\sim}{v},t)\, f_b(\underset{\sim}{x},\underset{\sim}{v},t)$$

$$- \left(\frac{m_a}{m} - 2\right) \frac{\partial f_a(\underset{\sim}{x},\underset{\sim}{v},t)}{\partial v_i} \quad \frac{\partial S_b(\underset{\sim}{x},\underset{\sim}{v},t)}{\partial v_i}$$

$$+ \frac{1}{2} \left(\frac{\partial^2 f_a(\underset{\sim}{x},\underset{\sim}{v},t)}{\partial v_i \;\partial v_j}\right) \left(\frac{\partial^2 T_b(\underset{\sim}{x},\underset{\sim}{v},t)}{\partial v_i \;\partial v_j}\right) \qquad (3.1)$$

where

$$A_a = \frac{4\pi Z_1^2 Z_2^2 e^4}{m_a^2} \,\ell n \left(\frac{2}{\theta_m}\right) \qquad (3.2)$$

$$m = \frac{m_a\, m_b}{m_a + m_b} \qquad (3.3)$$

$$S_b(\underset{\sim}{x},\underset{\sim}{v},t) = \int \frac{f_b(\underset{\sim}{x},\underset{\sim}{v}',t)}{|\underset{\sim}{v} - \underset{\sim}{v}'|}\, dv' \qquad (3.4)$$

$$T_b(\underset{\sim}{x},\underset{\sim}{v},t) = -\frac{1}{2\pi}\int \frac{S_b(\underset{\sim}{x},\underset{\sim}{v}',t)}{|v - v'|}\, d\underset{\sim}{v}' \qquad (3.5)$$

Here θ_m is obtained by setting the impact parameter equal to the Debye length (see Ref. 14, page 175). Our plasma has two components, so we let a and b run over the numbers 1 and 2, where 1 refers to -for example- the electrons, and 2 to the ions. Next, we introduce the spherical harmonic expansion of S_b and T_b

$$S_b(\underset{\sim}{x},\underset{\sim}{v},t) = \sum_{\ell m} S_{b,\ell m}(\underset{\sim}{x},v,t)\; Y_{\ell m}(w,\varphi) \qquad (3.6)$$

$$T_b(\underset{\sim}{x},\underset{\sim}{v},t) = \sum_{\ell m} T_{b,\ell m}(\underset{\sim}{x},v,t)\; Y_{\ell m}(w,\varphi) \qquad (3.7)$$

Also we separate into real and imaginary parts

$$S_{b,\ell m} = S^{+}_{b,\ell m} + i\, S^{-}_{b,\ell m} \qquad (3.8)$$

$$T_{b,\ell m} = T^{+}_{b,\ell m} + i\, T^{-}_{b,\ell m} \qquad (3.9)$$

We obtain

$$S^{\pm}_{b,\ell m}(\underset{\sim}{x},v,t) = \frac{4\pi}{2\ell + 1} \int_0^{\infty} g_{\ell}(v,v')\; f^{\pm}_{b,\ell m}(\underset{\sim}{x},v',t)v'^2\, dv' \qquad (3.10)$$

$$T^{\pm}_{b,\ell m}(\underset{\sim}{x},v,t) = -\frac{2}{2\ell+1}\int_0^\infty g_\ell(v,v')\, S^{\pm}_{b,\ell m}(\underset{\sim}{x},v',t)\, v'^2\, dv' \quad (3.11)$$

where

$$g_\ell(v,v') = \left\{\begin{array}{ll} \dfrac{(v')^\ell}{v^{\ell+1}} , & v > v' \\ \\ \dfrac{v^\ell}{(v')^{\ell+1}} , & v < v' \end{array}\right\} \quad (3.12)$$

Now, let us write (3.1) in terms of spherical velocity coordinates. First, we transform the factor $\frac{\partial f_a}{\partial v_i}\frac{\partial S_b}{\partial v_i}$ to spherical coordinates:

$$\frac{\partial f_a}{\partial v_i}\frac{\partial S_b}{\partial v_i} = A^{jk}\frac{\partial f_a}{\partial \xi_j}\frac{\partial S_b}{\partial \xi_k} \quad (3.13)$$

where

$$A^{jk} = \frac{\partial \xi_j}{\partial v_i}\frac{\partial \xi_k}{\partial v_i} \quad (3.14)$$

Using (2.27), we obtain

$$A^{jk} = \delta_{jk} A^j \quad (3.15)$$

where

$$A^1 = 1,\ A^2 = \frac{1 - w^2}{v^2},\ A^3 = \frac{1}{v^2(1 - w^2)} \tag{3.16}$$

Hence,

$$\frac{\partial f_a}{\partial v_i}\ \frac{\partial S_b}{\partial v_i} = A^j\ \frac{\partial f_a}{\partial \xi_j}\ \frac{\partial S_b}{\partial \xi_j} \tag{3.17}$$

Next, we consider the factor $\frac{\partial^2 f_a}{\partial v_i \partial v_j}\ \frac{\partial^2 T_b}{\partial v_i \partial v_j}$ which we write in the form

$$\begin{aligned} \frac{\partial^2 f_a}{\partial v_i \partial v_j}\ \frac{\partial^2 T_b}{\partial v_i \partial v_j} = {} & A^k A^\ell\ \frac{\partial^2 f_a}{\partial \xi_k \partial \xi_\ell}\ \frac{\partial^2 T_b}{\partial \xi_k \partial \xi_\ell} \\ & + B^{\ell k m}\left(\frac{\partial^2 f_a}{\partial \xi_\ell \partial \xi_k}\ \frac{\partial T_b}{\partial \xi_m} + \frac{\partial^2 T_b}{\partial \xi_\ell \partial \xi_k}\ \frac{\partial f_a}{\partial \xi_m}\right) \\ & + D^{k\ell}\ \frac{\partial f_a}{\partial \xi_k}\ \frac{\partial T_b}{\partial \xi_\ell} \end{aligned} \tag{3.18}$$

where A^k and A^ℓ are given explicitly by (3.16), and $B^{\ell k m}$ and $D^{k\ell}$ are defined by

$$B^{\ell k m} = \frac{\partial \xi_\ell}{\partial v_i}\ \frac{\partial \xi_k}{\partial v_j}\ \frac{\partial^2 \xi_m}{\partial v_i \partial v_j} \tag{3.19}$$

$$D^{k\ell} = \frac{\partial^2 \xi_k}{\partial v_i \partial v_j}\ \frac{\partial^2 \xi_\ell}{\partial v_i \partial v_j} \tag{3.20}$$

The first-order partial derivatives are given explicitly in (2.27). The 27 second-order partial derivatives are given explicitly as follows:

$$
\left.
\begin{array}{l}
\dfrac{\partial^2 v}{\partial v_1 \partial v_1} = \dfrac{w^2\cos^2\varphi + \sin^2\varphi}{v}, \quad \dfrac{\partial^2 v}{\partial v_1 \partial v_1} = \dfrac{w[(2-3w^2)\cos^2\varphi - \sin^2\varphi]}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_1 \partial v_1} = \dfrac{2\sin\varphi\cos\varphi}{v^2(1-w^2)} \\[2ex]
\dfrac{\partial^2 v}{\partial v_1 \partial v_2} = \dfrac{(w^2-1)\sin\varphi\cos\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_1 \partial v_2} = \dfrac{3w(1-w^2)\sin\varphi\cos\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_1 \partial v_2} = \dfrac{\sin^2\varphi - \cos^2\varphi}{v^2(1-w^2)} \\[2ex]
\dfrac{\partial^2 v}{\partial v_1 \partial v_3} = \dfrac{-w\sqrt{1-w^2}\cos\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_1 \partial v_3} = \dfrac{(3w^2-1)\sqrt{1-w^2}\cos\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_1 \partial v_3} = 0 \\[2ex]
\dfrac{\partial^2 v}{\partial v_2 \partial v_1} = \dfrac{(w^2-1)\sin\varphi\cos\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_2 \partial v_1} = \dfrac{3w(1-w^2)\sin\varphi\cos\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_2 \partial v_1} = \dfrac{\sin^2\varphi - \cos^2\varphi}{v^2(1-w^2)} \\[2ex]
\dfrac{\partial^2 v}{\partial v_2 \partial v_2} = \dfrac{w^2\sin^2\varphi + \cos^2\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_2 \partial v_2} = w[(2-3w^2)\sin^2\varphi - \cos^2\varphi], \quad \dfrac{\partial^2 \varphi}{\partial v_2 \partial v_2} = \dfrac{-2\sin\varphi\cos\varphi}{v^2(1-w^2)} \\[2ex]
\dfrac{\partial^2 v}{\partial v_2 \partial v_3} = \dfrac{-w\sqrt{1-w^2}\sin\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_2 \partial v_3} = \dfrac{(3w^2-1)\sqrt{1-w^2}\sin\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_2 \partial v_3} = 0 \\[2ex]
\dfrac{\partial^2 v}{\partial v_3 \partial v_1} = \dfrac{-w\sqrt{1-w^2}\cos\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_3 \partial v_1} = \dfrac{(3w^2-1)\sqrt{1-w^2}\cos\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_3 \partial v_1} = 0 \\[2ex]
\dfrac{\partial^2 v}{\partial v_3 \partial v_2} = \dfrac{-w\sqrt{1-w^2}\sin\varphi}{v}, \quad \dfrac{\partial^2 w}{\partial v_3 \partial v_2} = \dfrac{(3w^2-1)\sqrt{1-w^2}\sin\varphi}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_3 \partial v_2} = 0 \\[2ex]
\dfrac{\partial^2 v}{\partial v_3 \partial v_3} = \dfrac{1-w^2}{v}, \quad \dfrac{\partial^2 w}{\partial v_3 \partial v_3} = \dfrac{3w(w^2-1)}{v^2}, \quad \dfrac{\partial^2 \varphi}{\partial v_3 \partial v_3} = 0
\end{array}
\right\} \quad (3.21)
$$

By using (2.27) and (3.21), we now obtain explicit expressions for $B^{\ell km}$ and $D^{k\ell}$:

$$
\left.
\begin{array}{lll}
B^{111} = 0 & B^{112} = 0 & B^{113} = 0 \\
B^{121} = 0 & B^{122} = \dfrac{w^2-1}{v^3} & B^{123} = 0 \\
B^{131} = 0 & B^{132} = 0 & B^{133} = \dfrac{1}{v^3(w^2-1)} \\
B^{211} = 0 & B^{212} = \dfrac{w^2-1}{v^3} & B^{213} = 0 \\
B^{221} = \dfrac{1-w^2}{v^3} & B^{222} = \dfrac{w(w^2-1)}{v^4} & B^{223} = 0 \\
B^{231} = 0 & B^{232} = 0 & B^{233} = \dfrac{w}{v^4(1-w^2)} \\
B^{311} = 0 & B^{312} = 0 & B^{313} = \dfrac{1}{v^3(w^2-1)} \\
B^{321} = 0 & B^{322} = 0 & B^{323} = \dfrac{w}{v^4(1-w^2)} \\
B^{331} = \dfrac{1}{v^3(1-w^2)} & B^{332} = \dfrac{w}{v^4(w^2-1)} & B^{333} = 0
\end{array}
\right\} \quad (3.22)
$$

$$
\left.
\begin{array}{lll}
D^{11} = \dfrac{2}{v^2} & D^{12} = \dfrac{-2w}{v^3} & D^{13} = 0 \\
D^{21} = \dfrac{-2w}{v^3} & D^{22} = \dfrac{2}{v^4} & D^{23} = 0 \\
D^{31} = 0 & D^{32} = 0 & D^{33} = \dfrac{2}{v^4(1-w^2)^2}
\end{array}
\right\} \quad (3.23)
$$

Using (3.15) through (3.23) and (2.27), we write (3.1) explicitly as follows:

$$\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab} = 4\pi\left(\frac{m_a}{m}-1\right)f_a f_b$$

$$-\left(\frac{m_a}{m}-2\right)\left[\frac{\partial f_a}{\partial v}\frac{\partial S_b}{\partial v}+\frac{(1-w^2)}{v^2}\frac{\partial f_a}{\partial w}\frac{\partial S_b}{\partial w}+\frac{1}{v^2(1-w^2)}\frac{\partial f_a}{\partial \varphi}\frac{\partial S_b}{\partial \varphi}\right]$$

$$+\frac{1}{2}\left[\frac{\partial^2 f_a}{\partial v^2}\frac{\partial^2 T_b}{\partial v^2}+\frac{(1-w^2)^2}{v^4}\frac{\partial^2 f_a}{\partial w^2}\frac{\partial^2 T_b}{\partial w^2}+\frac{1}{v^4(1-w^2)^2}\frac{\partial^2 f_a}{\partial \varphi^2}\frac{\partial^2 T_b}{\partial \varphi^2}\right.$$

$$+\frac{2(1-w^2)}{v^2}\frac{\partial^2 f_a}{\partial v\partial w}\frac{\partial^2 T_b}{\partial v\partial w}+\frac{2}{v^2(1-w^2)}\frac{\partial^2 f_a}{\partial v\partial \varphi}\frac{\partial^2 T_b}{\partial v\partial \varphi}+\frac{2}{v^4}\frac{\partial^2 f_a}{\partial w\partial \varphi}\frac{\partial^2 T_b}{\partial w\partial \varphi}$$

$$+\frac{(1-w^2)}{v^3}\left(\frac{\partial^2 f_a}{\partial w^2}\frac{\partial T_b}{\partial v}+\frac{\partial^2 T_b}{\partial w^2}\frac{\partial f_a}{\partial v}\right)+\frac{1}{v^3(1-w^2)}\left(\frac{\partial^2 f_a}{\partial \varphi^2}\frac{\partial T_b}{\partial v}+\frac{\partial^2 T_b}{\partial \varphi^2}\frac{\partial f_a}{\partial v}\right)$$

$$+\frac{2(w^2-1)}{v^3}\left(\frac{\partial^2 f_a}{\partial v\partial w}\frac{\partial T_b}{\partial w}+\frac{\partial^2 T_b}{\partial v\partial w}\frac{\partial f_a}{\partial w}\right)+\frac{w(w^2-1)}{v^4}\left(\frac{\partial^2 f_a}{\partial w^2}\frac{\partial T_b}{\partial w}+\frac{\partial^2 T_b}{\partial w^2}\frac{\partial f_a}{\partial w}\right)$$

$$+\frac{w}{v^4(w^2-1)}\left(\frac{\partial^2 f_a}{\partial \varphi^2}\frac{\partial T_b}{\partial w}+\frac{\partial^2 T_b}{\partial \varphi^2}\frac{\partial f_a}{\partial w}\right)+\frac{2}{v^3(w^2-1)}\left(\frac{\partial^2 f_a}{\partial v\partial \varphi}\frac{\partial T_b}{\partial \varphi}+\frac{\partial^2 T_b}{\partial v\partial \varphi}\frac{\partial f_a}{\partial \varphi}\right)$$

$$+\frac{2w}{v^4(1-w^2)}\left(\frac{\partial^2 f_a}{\partial w\partial \varphi}\frac{\partial T_b}{\partial \varphi}+\frac{\partial^2 T_b}{\partial w\partial \varphi}\frac{\partial f_a}{\partial \varphi}\right)+\frac{2}{v^2}\frac{\partial f_a}{\partial v}\frac{\partial T_b}{\partial v}$$

$$\left.-\frac{2w}{v^3}\left(\frac{\partial f_a}{\partial v}\frac{\partial T_b}{\partial w}+\frac{\partial f_a}{\partial w}\frac{\partial T_b}{\partial v}\right)+\frac{2}{v^4}\frac{\partial f_a}{\partial w}\frac{\partial T_b}{\partial w}+\frac{2}{v^4(1-w^2)^2}\frac{\partial f_a}{\partial \varphi}\frac{\partial T_b}{\partial \varphi}\right] \qquad (3.24)$$

When the collisions involve like particles, the second term drops out, if that is any consolation. There is one simplifying feature, however. None of the coefficients in (3.24) depends on φ. This will greatly simplify the spherical harmonic integrals.

Next, we substitute the spherical harmonic expansions (2.30), (3.6) and (3.7) into (3.24), multiply on the left by $Y^*_{\ell m}$, and integrate over the solid angle $d\Omega = dw d\varphi$. We obtain

$$\left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right)_{\ell m} = \left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right)^+_{\ell m} + i\left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right)^-_{\ell m} \tag{3.25}$$

where

$$\begin{aligned}\left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right)^{\pm} = \sum_{\substack{\ell' m' \\ \ell'' m''}} \Bigg\{ & 4\pi\left(\frac{m_a}{m}-1\right) {}_1I^{\ell\ell'\ell''}_{mm'm''}\left(f^+_{a,\ell'm'}\; f^{\pm}_{b,\ell''m''} \mp f^{\mp}_{b,\ell'm'}\; f^-_{b,\ell''m''}\right) \\
& -\left(\frac{m_a}{m}-2\right)\Bigg[{}_1I^{\ell\ell'\ell''}_{mm'm''}\left(\frac{\partial f^+_{a,\ell'm'}}{\partial v}\;\frac{\partial S^{\pm}_{b,\ell''m''}}{\partial v} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\;\frac{\partial S^-_{b,\ell''m''}}{\partial v}\right) \\
& + \frac{{}_2I^{\ell\ell'\ell''}_{mm'm''}}{v^2}\left(f^+_{a,\ell'm'}\; S^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\; S^-_{b,\ell''m''}\right) \\
& + \frac{{}_3I^{\ell\ell'\ell''}_{mm'm''}}{v^2}\left(f^+_{a,\ell'm'}\; S^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\; S^-_{b,\ell''m''}\right)\Bigg] \\
& + \tfrac{1}{2}\Bigg[{}_1I^{\ell\ell'\ell''}_{mm'm''}\left(\frac{\partial^2 f^+_{a,\ell'm'}}{\partial v^2}\;\frac{\partial^2 T^{\pm}_{b,\ell''m''}}{\partial v^2} \mp \frac{\partial^2 f^{\mp}_{a,\ell'm'}}{\partial v^2}\;\frac{\partial^2 T^-_{b,\ell''m''}}{\partial v^2}\right)\end{aligned}$$

$$+ \frac{{}_4I^{\ell\ell'\ell''}_{mm'm''}}{v^4}\left(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''}\right)$$

$$+ \frac{{}_5I^{\ell\ell'\ell''}_{mm'm''}}{v^4}\left(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''}\right)$$

$$+ \frac{{}^2{}_2I^{\ell\ell'\ell''}_{mm'm''}}{v^2}\left(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\,\frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\,\frac{\partial T^{-}_{b,\ell''m''}}{\partial v}\right)$$

$$+ \frac{{}^2{}_3I^{\ell\ell'\ell''}_{mm'm''}}{v^2}\left(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\,\frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\,\frac{\partial T^{-}_{b,\ell''m''}}{\partial v}\right)$$

$$+ \frac{{}^2{}_6I^{\ell\ell'\ell''}_{mm'm''}}{v^4}\left(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''}\right)$$

$$+ \frac{{}_7I^{\ell\ell'\ell''}_{mm'm''}}{v^3}\left(f^{+}_{a,\ell'm'}\,\frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} + T^{+}_{b,\ell'm'}\,\frac{\partial f^{\pm}_{a,\ell''m''}}{\partial v} \mp f^{\mp}_{a,\ell'm'}\,\frac{\partial T^{-}_{b,\ell''m''}}{\partial v}\right.$$

$$\left. \mp T^{\mp}_{b,\ell'm'}\,\frac{\partial f^{-}_{a,\ell''m''}}{\partial v}\right) + \frac{{}_8I^{\ell\ell'\ell''}_{mm'm''}}{v^3}\left(f^{+}_{a,\ell'm'}\,\frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} + T^{+}_{b,\ell'm'}\,\frac{\partial f^{\pm}_{a,\ell''m''}}{\partial v}\right.$$

$$\left. \mp f^{\mp}_{a,\ell'm'}\,\frac{\partial T^{-}_{b,\ell''m''}}{\partial v} \mp T^{\mp}_{b,\ell'm'}\,\frac{\partial f^{-}_{a,\ell''m''}}{\partial v}\right) - \frac{{}^2{}_2I^{\ell\ell'\ell''}_{mm'm''}}{v^3}\left(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\, T^{\pm}_{b,\ell''m''}\right.$$

$$\left. + \frac{\partial T^{+}_{b,\ell'm'}}{\partial v}\, f^{\pm}_{a,\ell''m''} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\, T^{-}_{b,\ell''m''} \mp \frac{\partial T^{\mp}_{b,\ell'm'}}{\partial v}\, f^{-}_{a,\ell''m''}\right)$$

$$
+ \frac{{}_{9}I^{\ell\ell'\ell''}_{mm'm''}}{v^{4}} \Bigg(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} + T^{+}_{b,\ell'm'}\, f^{\pm}_{a,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''}
$$

$$
\mp T^{\mp}_{b,\ell'm'}\, f^{-}_{a,\ell''m''} \Bigg) + \frac{{}_{10}I^{\ell\ell'\ell''}_{mm'm''}}{v^{4}} \Bigg(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} + T^{+}_{b,\ell'm'}\, f^{\pm}_{a,\ell''m''}
$$

$$
\mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''} \mp T^{\mp}_{b,\ell'm'}\, f^{-}_{a,\ell''m''} \Bigg) - \frac{{}^{2}_{3}I^{\ell\ell'\ell''}_{mm'm''}}{v^{3}} \Bigg(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\, T^{\pm}_{b,\ell''m''}
$$

$$
+ \frac{\partial T^{+}_{b,\ell'm'}}{\partial v}\, f^{\pm}_{a,\ell''m''} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\, T^{-}_{b,\ell''m''} \mp \frac{\partial T^{\mp}_{b,\ell'm'}}{\partial v}\, f^{-}_{a,\ell''m''} \Bigg)
$$

$$
+ \frac{{}^{2}_{11}I^{\ell\ell'\ell''}_{mm'm''}}{v^{4}} \Bigg(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} + T^{+}_{b,\ell'm'}\, f^{\pm}_{a,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''}
$$

$$
\mp T^{\mp}_{b,\ell'm'}\, f^{-}_{a,\ell''m''} \Bigg) + \frac{{}^{2}_{1}I^{\ell\ell'\ell''}_{mm'm''}}{v^{2}} \Bigg(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\, \frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\, \frac{\partial T^{-}_{b,\ell''m''}}{\partial v} \Bigg)
$$

$$
- \frac{{}^{2}_{12}I^{\ell\ell'\ell''}_{mm'm''}}{v^{3}} \Bigg(\frac{\partial f^{+}_{a,\ell'm'}}{\partial v}\, T^{\pm}_{b,\ell''m''} \mp \frac{\partial f^{\mp}_{a,\ell'm'}}{\partial v}\, T^{-}_{b,\ell''m''} \Bigg)
$$

$$
- \frac{{}^{2}_{13}I^{\ell\ell'\ell''}_{mm'm''}}{v^{3}} \Bigg(f^{+}_{a,\ell'm'}\, \frac{\partial T^{\pm}_{b,\ell''m''}}{\partial v} \mp f^{\mp}_{a,\ell'm'}\, \frac{\partial T^{-}_{b,\ell''m''}}{\partial v} \Bigg)
$$

$$
+ \frac{{}^{2}_{14}I^{\ell\ell'\ell''}_{mm'm''}}{v^{4}} \Bigg(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''} \Bigg)
$$

$$
+ \frac{{}^{2}_{15}I^{\ell\ell'\ell''}_{mm'm''}}{v^{4}} \Bigg(f^{+}_{a,\ell'm'}\, T^{\pm}_{b,\ell''m''} \mp f^{\mp}_{a,\ell'm'}\, T^{-}_{b,\ell''m''} \Bigg) \Bigg] \Bigg\} \qquad (3.26)
$$

$$\left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right)^{\pm}_{\ell m} = \begin{pmatrix}\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}\int Y^*_{\ell m}\left(\frac{1}{A_a}\left.\frac{\partial f_a}{\partial t}\right|_{ab}\right) d\Omega \tag{3.27}$$

$$\left.\begin{aligned}
{}_1I^{\ell\ell'\ell''}_{mm'm''} &= \int Y^*_{\ell m}\, Y_{\ell' m'}\, Y_{\ell'' m''}\, d\Omega \\
{}_2I^{\ell\ell'\ell''}_{mm'm''} &= \int (1-w^2)\, Y^*_{\ell m}\, \frac{\partial Y_{\ell' m'}}{\partial w}\, \frac{\partial Y_{\ell'' m''}}{\partial w}\, d\Omega \\
{}_3I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{1}{1-w^2}\, Y^*_{\ell m}\, \frac{\partial Y_{\ell' m'}}{\partial \varphi}\, \frac{\partial Y_{\ell'' m''}}{\partial \varphi}\, d\Omega \\
{}_4I^{\ell\ell'\ell''}_{mm'm''} &= \int (1-w^2)^2\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial w^2}\, \frac{\partial^2 Y_{\ell'' m''}}{\partial w^2}\, d\Omega \\
{}_5I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{1}{(1-w^2)^2}\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial \varphi^2}\, \frac{\partial^2 Y_{\ell'' m''}}{\partial \varphi^2}\, d\Omega \\
{}_6I^{\ell\ell'\ell''}_{mm'm''} &= \int Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial w \partial \varphi}\, \frac{\partial^2 Y_{\ell'' m''}}{\partial w \partial \varphi}\, d\Omega \\
{}_7I^{\ell\ell'\ell''}_{mm'm''} &= \int (1-w^2)\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial w^2}\, Y_{\ell'' m''}\, d\Omega \\
{}_8I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{1}{(1-w^2)}\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial \varphi^2}\, Y_{\ell'' m''}\, d\Omega \\
{}_9I^{\ell\ell'\ell''}_{mm'm''} &= \int w(w^2-1)\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial w^2}\, \frac{\partial Y_{\ell'' m''}}{\partial w}\, d\Omega \\
{}_{10}I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{w}{w^2-1}\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell' m'}}{\partial \varphi^2}\, \frac{\partial Y_{\ell'' m''}}{\partial w}\, d\Omega
\end{aligned}\right\} \tag{3.28}$$

$$
\left.
\begin{aligned}
{}_{11}I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{w}{1-w^2}\, Y^*_{\ell m}\, \frac{\partial^2 Y_{\ell'm'}}{\partial w\partial\varphi}\, \frac{\partial Y_{\ell''m''}}{\partial\varphi}\, d\Omega \\
{}_{12}I^{\ell\ell'\ell''}_{mm'm''} &= \int w\, Y^*_{\ell m}\, Y_{\ell'm'}\, \frac{\partial Y_{\ell''m''}}{\partial w}\, d\Omega \\
{}_{13}I^{\ell\ell'\ell''}_{mm'm''} &= \int w\, Y^*_{\ell m}\, \frac{\partial Y_{\ell'm'}}{\partial w}\, Y_{\ell''m''}\, d\Omega \\
{}_{14}I^{\ell\ell'\ell''}_{mm'm''} &= \int Y^*_{\ell m}\, \frac{\partial Y_{\ell'm'}}{\partial w}\, \frac{\partial Y_{\ell''m''}}{\partial w}\, d\Omega \\
{}_{15}I^{\ell\ell'\ell''}_{mm'm''} &= \int \frac{1}{(1-w^2)^2}\, Y^*_{\ell m}\, \frac{\partial Y_{\ell'm'}}{\partial\varphi}\, \frac{\partial Y_{\ell''m''}}{\partial\varphi}\, d\Omega
\end{aligned}
\right\}
$$

These integrals are all real. In equating the transport term to the collision term, we must remember to multiply the expressions in (3.26) by A_a. We now have a rigorous reduction of the Fokker-Planck equation. We have not indicated explicitly the various δ-functions and vanishing integrals that are obtained when we carry out (3.28), but this is best taken care of by doing the algebra of these integral coefficients automatically on a computer. The problem is still quite formidable for general three-dimensional configuration space dependence. However, there are many interesting problems which have a high degree of configuration space symmetry. An example is given in Section V.

IV. THE SELF-CONSISTENCY CONDITIONS

In order to discuss the self-consistency conditions, we must first define some velocity-integrated quantities. The space density of particles is defined by

$$
N_a(\underset{\sim}{x},t) = \int f_a(\underset{\sim}{x},\underset{\sim}{v},t)\, d\underset{\sim}{v} \tag{4.1}
$$

Substituting the expansion (2.30) into (4.1), we obtain

$$N_a(\underset{\sim}{x},t) = \sqrt{4\pi} \int_0^\infty f^+_{a,oo} (\underset{\sim}{x},v,t)\, v^2\, dv \tag{4.2}$$

The average velocity is defined by

$$\left\langle V_i \right\rangle_a = \frac{1}{N_a(\underset{\sim}{x},t)} \int V_i f_a(\underset{\sim}{x},\underset{\sim}{v},t)\, d\underset{\sim}{v} \tag{4.3}$$

Making use of the separation in (2.6), we obtain

$$\left\langle V_i \right\rangle_a = u_{a,i} + \left\langle v_i \right\rangle_a \tag{4.4}$$

where

$$\left\langle v_i \right\rangle_a = \frac{1}{N_a(x,t)} \int v_i f_a(\underset{\sim}{x},\underset{\sim}{v},t)\, d\underset{\sim}{v} \tag{4.5}$$

Substituting the expansion (2.30) into (4.5), we obtain

$$\left.\begin{aligned}
\left\langle v_1 \right\rangle_a &= \frac{1}{N_a} \sum_{\ell} \Bigg[I_{\ell 1} \int_0^\infty f^+_{a,\ell 1} (\underset{\sim}{x},v,t)\, v^3\, dv \\
&\qquad + I_{\ell,-1} \int_0^\infty f^+_{a,\ell,-1} (\underset{\sim}{x},v,t)\, v^3\, dv \Bigg] \\
\left\langle v_2 \right\rangle_a &= -\frac{1}{N_a} \sum_{\ell} \Bigg[I_{\ell 1} \int_0^\infty f^-_{a,\ell 1} (\underset{\sim}{x},v,t)\, v^3\, dv \\
&\qquad - I_{\ell,-1} \int_0^\infty f^-_{a,\ell,-1} (\underset{\sim}{x},v,t)\, v^3\, dv \Bigg] \\
\left\langle v_3 \right\rangle_a &= \frac{1}{N_a} \sum_{\ell} I_{\ell o} \int_0^\infty f^+_{a,\ell o} (\underset{\sim}{x},v,t)\, v^3\, dv
\end{aligned}\right\} \tag{4.6}$$

where

$$\left.\begin{aligned} I_{\ell,\pm 1} &= \pi a_{\ell,\pm 1}\int \sqrt{1-w^2}\, P_\ell^{\pm 1}\, dw \\ I_{\ell o} &= 2\pi a_{\ell o}\int w\, P_\ell^{o}\, dw \end{aligned}\right\} \qquad (4.7)$$

The $a_{\ell m}$ are given by (2.36).

In the present formalism, $\underset{\sim}{u}_a(\underset{\sim}{x},t)$ is arbitrary. If we set $\underset{\sim}{u}_a = 0$, then our distribution is given in terms of the absolute velocity. If we want $\underset{\sim}{u}_a$ to be the average velocity according to (2.7), then we must include the self-consistency condition,

$$\left\langle \underset{\sim}{v}_a \right\rangle = 0 \qquad (4.8)$$

This is optional, of course.

Next, we write the mass and charge densities:

$$\rho_a(\underset{\sim}{x},t) = m_a N_a(\underset{\sim}{x},t) \qquad (4.9)$$

$$\rho_{ea}(\underset{\sim}{x},t) = q_a N_a(\underset{\sim}{x},t) \qquad (4.10)$$

The current density in emu is given by

$$\underset{\sim}{J}_a(\underset{\sim}{x},t) = \frac{\rho_{ea}(\underset{\sim}{u}_a + \left\langle \underset{\sim}{v} \right\rangle_a)}{c} \qquad (4.11)$$

For the total quantities, we sum over components. Thus,

$$N(\underset{\sim}{x},t) = \sum_a N_a(\underset{\sim}{x},t) \qquad (4.12)$$

$$\rho(\underset{\sim}{x},t) = \sum_a \rho_a(\underset{\sim}{x},t) \qquad (4.13)$$

$$\rho_e(\underset{\sim}{x},t) = \sum_a \rho_{ea}(\underset{\sim}{x},t) \tag{4.14}$$

$$\underset{\sim}{J}(\underset{\sim}{x},t) = \sum_a \underset{\sim}{J}_a(\underset{\sim}{x},t) \tag{4.15}$$

Using (4.14) and (4.15) for the appropriate source densities, we solve Maxwell's equations,

$$\nabla \cdot \underset{\sim}{E} = 4\pi\rho_e \tag{4.16}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \tag{4.17}$$

$$\nabla \times \underset{\sim}{B} = 4\pi\underset{\sim}{J} + \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} \tag{4.18}$$

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} \tag{4.19}$$

The self-consistency condition on the solutions $\underset{\sim}{E}$ and $\underset{\sim}{B}$ is that they agree with the fields appearing in (2.33).

V. THETA-PINCH GEOMETRY

As an example of how to apply the above method to a real physical problem, we consider the plasma dynamics of the theta-pinch in one configuration-space variable. We assume axial symmetry so that only the radial independent variable ($\eta_3 \equiv r$) occurs in the problem. Because the average velocities will be in the radial direction, we choose our velocity spherical harmonic axis to be along the radial configuration-space direction. Therefore, we make the correspondence,

$$\left.\begin{aligned} \hat{\underset{\sim}{v}}_1 &= \hat{\underset{\sim}{\epsilon}}_1 = \hat{\underset{\sim}{\theta}} \\ \hat{\underset{\sim}{v}}_2 &= \hat{\underset{\sim}{\epsilon}}_2 = \hat{\underset{\sim}{z}} \\ \hat{\underset{\sim}{v}}_3 &= \hat{\underset{\sim}{\epsilon}}_3 = \hat{\underset{\sim}{r}} \end{aligned}\right\} \tag{5.1}$$

Inasmuch as charge separation will occur only in the radial direction, we will have an electric field only in the radial direction:

$$\underset{\sim}{E} = \hat{\underset{\sim}{\epsilon}}_3 \, E_3(r,t) \tag{5.2}$$

The primary magnetic field under consideration, which is in the $\hat{\underset{\sim}{z}}$-direction, produces the theta-pinch and penetrates to a certain extent into the confined plasma. However, the electric field defined in (5.2) gives rise to a radial current density. This will produce components of $\underset{\sim}{B}$ in the $\hat{\underset{\sim}{\theta}}$ and $\hat{\underset{\sim}{z}}$ directions. Therefore, we have

$$\underset{\sim}{B} = \hat{\underset{\sim}{\epsilon}}_1 B_1(r,t) + \hat{\underset{\sim}{\epsilon}}_2 B_2(r,t) \tag{5.3}$$

The effective field $\underset{\sim}{E}^*$ appearing in (2.33) is given according to (2.29) by

$$\underset{\sim}{E}^* = \underset{\sim}{E} + \frac{\underset{\sim}{u} \times \underset{\sim}{B}}{c} \tag{5.4}$$

Using (5.2), (5.3), and (5.4) and the fact that the average velocities will be in the radial direction, we obtain

$$\underset{\sim}{E}^* = \hat{\underset{\sim}{\epsilon}}_1 \left(- \frac{u_3 B_2}{c} \right) + \hat{\underset{\sim}{\epsilon}}_2 \left(\frac{u_3 B_1}{c} \right) + \hat{\underset{\sim}{\epsilon}}_3 \, E_3 \tag{5.5}$$

Thus, we see that all three components in (5.5) are non-vanishing, even in this simple geometry.

For this problem, then, (2.33) reduces to the following equation:

$$
\begin{aligned}
(\delta f)^{\pm}_{\ell m} = \frac{Df^{\pm}_{\ell m}}{Dt} + \sum_{\ell'} \Bigg[& v\left({}_2I^{\ell' m}_{\ell m} \frac{\partial f^{\pm}_{\ell' m}}{\partial r}\right) \\
& + \frac{qE^*_1}{m}\left({}_1I^{\ell', m+1}_{\ell m} \frac{\partial f^{\pm}_{\ell', m+1}}{\partial v} + {}_1I^{\ell', m-1}_{\ell m} \frac{\partial f^{\pm}_{\ell', m-1}}{\partial v}\right) \\
& \mp \frac{qE^*_2}{m}\left({}_1I^{\ell', m+1}_{\ell m} \frac{\partial f^{\mp}_{\ell', m+1}}{\partial v} - {}_1I^{\ell', m-1}_{\ell m} \frac{\partial f^{\mp}_{\ell', m-1}}{\partial v}\right) \\
& + \left(\frac{qE^*_3}{m} - \frac{Du_3}{Dt}\right)\left({}_2I^{\ell' m}_{\ell m} \frac{\partial f^{\pm}_{\ell' m}}{\partial v}\right) \\
& - \frac{qE^*_1}{mv^2}\left({}_3I^{\ell', m+1}_{\ell m} f^{\pm}_{\ell', m+1} + {}_3I^{\ell', m-1}_{\ell m} f^{\pm}_{\ell', m-1}\right) \\
& \pm \frac{qE^*_2}{mv^2}\left({}_3I^{\ell', m+1}_{\ell m} f^{\mp}_{\ell', m+1} - {}_3I^{\ell', m-1}_{\ell m} f^{\mp}_{\ell', m-1}\right) \\
& + \left(\frac{qE^*_3}{m} - \frac{Du_3}{Dt}\right)\frac{1}{v^2}\left({}_4I^{\ell' m}_{\ell m} f^{\pm}_{\ell' m}\right) \\
& + \frac{qE^*_1}{mv^2}\left({}_6I^{\ell', m+1}_{\ell m} f^{\pm}_{\ell', m+1} - {}_6I^{\ell', m-1}_{\ell m} f^{\pm}_{\ell', m-1}\right) \\
& \mp \frac{qE^*_2}{mv^2}\left({}_6I^{\ell', m+1}_{\ell m} f^{\mp}_{\ell', m+1} + {}_6I^{\ell', m-1}_{\ell m} f^{\mp}_{\ell', m-1}\right) \\
& \pm \frac{qB_1}{mcv}\left({}_5I^{\ell', m+1}_{\ell m} f^{\mp}_{\ell', m+1} - {}_5I^{\ell', m-1}_{\ell m} f^{\mp}_{\ell', m-1}\right) \\
& + \frac{qB_2}{mcv}\left({}_5I^{\ell', m+1}_{\ell m} f^{\pm}_{\ell', m+1} + {}_5I^{\ell', m-1}_{\ell m} f^{\pm}_{\ell', m-1}\right) \\
& \mp \frac{qB_1}{mcv}\left({}_7I^{\ell', m+1}_{\ell m} f^{\mp}_{\ell', m+1} + {}_7I^{\ell', m-1}_{\ell m} f^{\mp}_{\ell', m-1}\right) \\
& - \frac{qB_2}{mcv}\left({}_7I^{\ell', m+1}_{\ell m} f^{\pm}_{\ell', m+1} - {}_7I^{\ell', m-1}_{\ell m} f^{\pm}_{\ell', m-1}\right) \\
& - v\frac{\partial u_3}{\partial r}\left({}_{11}I^{\ell' m}_{\ell m} \frac{\partial f^{\pm}_{\ell' m}}{\partial v}\right) - \frac{4}{v}\frac{\partial u_3}{\partial r}\left({}_{12}I^{\ell' m}_{\ell m} f^{\pm}_{\ell' m}\right)\Bigg] \qquad (5.6)
\end{aligned}
$$

Again, we have used the fact that the average velocity is in the radial direction. The expression (2.18) for D/Dt reduces in the present case to

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_3 \frac{\partial}{\partial r} \tag{5.7}$$

The collision term discussed in Section III will be the same for all configuration-space geometries. In principle, (5.6), coupled with the collision terms, is sufficient to do a good job of representing the dynamics of the theta-pinch, both in the equilibrium and non-equilibrium situations. However, we should point out one complicating feature. With the electric fields and charge separation in the problem, the equations will give a detailed treatment of phenomena such as plasma oscillations that occur in times short compared to the times required for the main theta-pinch dynamics. Therefore, a machine program will spend much time calculating details that are not too important in the main problem. Thus, we may anticipate a calculation of excessive length. It is likely that preliminary simple calculations will have to be done with (5.6) in order to learn how to unscramble the various relatively different types of phenomena.

VI. GENERAL COMMENTS

We have outlined a difficult application in the previous section. However, our equations can form a point of departure for a multitude of more simple, but still interesting, problems. The effects of charge separation on shock wave propagation in a plasma have been recently investigated in the MHD approximation by Jaffrin and Probstein.[15] It may be of interest to apply transport theory in a similar investigation.

Another property of this method is that it forms the basis of a separation of departures from the Maxwell-Boltzmann distribution into two types: 1) departures of the dependence on the velocity magnitude, and

2) departures from isotropy. In the first case, we keep only the Y_{oo} terms but calculate f_{oo} from the resulting integro-partial differential equations. In the second case, we assume a form for $f^{\pm}_{\ell m}$ and do a spherical harmonic analysis of the anisotropies. For simplicity, we may do such problems with a uniform space-dependence.

We can investigate the transport coefficients in the near-equilibrium region. In particular, we would like to determine the resistivity in the transition region between weak and strong magnetic fields. Isolated investigations of these various phenomena may allow significant improvement in our MHD treatment of the theta-pinch.

REFERENCES

1. K. Hain and A. C. Kolb, Nuclear Fusion, 1962 Supplement, Part 2, 561.

2. T. A. Oliphant, Los Alamos Scientific Laboratory Report LAMS-2944 (1963)

3. T. A. Oliphant and F. L. Ribe, Bull. Am. Phys. Soc.[2] 10, 224, article P5 (1965).

4. M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).

5. M. Krook, Am. J. Phys. (N.Y.) 6, 188 (1959).

6. C. K. Chu, Phys. Fluids 8, 12 (1965).

7. H. Salwen, C. E. Goosch, and S. Ziering, Phys. Fluids 7, 180 (1963).

8. H. M. Mott-Smith, Phys. Rev. 82, 885 (1951).

9. H. W. Liepmann, R. Narasimha, and M. T. Chahine, Phys. Fluids 5, 1313 (1962).

10. B. Alder, S. Fernbach, M. Rotenberg, Eds., Methods in Computational Physics (Academic Press, New York, 1965), Vol. 4, pp. 83-107 by M. T. Chahine.

11. S. Chapmann and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, (Cambridge University Press, London, 1952) 2nd Edition.

12. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (McGraw-Hill Publishing Co., Inc., New York 1953).

13. E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra, (Cambridge University Press, London, 1935).

14. C. L. Longmire, Elementary Plasma Physics, (Interscience Publishers, Inc., New York, 1963).

15. M. Y. Jaffrin and R. F. Probstein, Phys. Fluids 7, 1658 (1964).