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Energy Loss by Nuclear Elastic Scattering

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by

Joseph J. Devaney
Myron L. Stein

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ENERGY LOSS BY NUCLEAR ELASTIC SCATTERING

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ABSTRACT

The rate of energy loss suffered by a heavy charged particle from nuclear forces elastic scattering, plus nuclear Coulomb interference, is derived, and the rate for protons, deuterons, and alphas into deuterium is given as a function of incident energy.

I. Introduction. In the problem of energy loss of heavy charged particles at low energies in matter, one must consider the loss of energy of the particle to electrons by Coulomb interaction, to nuclei by Coulomb and hadronic elastic scattering (plus their interference), to nuclei by nuclear reactions, and by nuclear bremsstrahlung (small). An easy computation of the energy loss per unit path from the first two processes is afforded, for example, by the work of Longmire.[1] The present paper gives the energy loss from the next two processes in the same form and with the same assumption as Longmire's work. Namely, we offer formulas and a few selected computations for the energy loss of a charged particle due to nuclear force elastic scattering plus nuclear Coulomb interference. The particle-to-electron and particle-to-nucleus pure Coulomb losses are to be found in Longmire.

We deem it not appropriate to include the specifically nuclear Coulomb energy loss in our examples, for its magnitude depends on the assumed electron temperature and density. However, the nuclear forces-nuclear Coulomb interference term is integrable without need of a cutoff and thus is not so dependent. It is therefore included.

It should be noted that the Coulomb scattering of the charged particle by both electrons and nuclei is strongly peaked in the forward direction so that energy loss per path length is a useful description. However, hadronic scattering of unlike particles may even be peaked in the backward direction so that its contribution leads to a more erratic path. Nonetheless, energy deposition or loss per path length was judged to be relevant because such losses are in the same format as Coulombic losses, are relatively easy to obtain, and are model independent. Should hadronic energy deposition be dominant, one may wish to consider calculations such as the Monte Carlo types in order to follow energy deposition more precisely.

For identical particles we choose to follow the most energetic resultant particle so that the maximum laboratory scattering angle per collision will be 45° and we will thus give the average energy loss as a function of path of the most energetic remainder of each collision.

In Longmire's formula (9-58), p. 203, the approximations are equivalent to temperature, $T = 0$,

[1] C. L. Longmire, Elementary Plasma Physics, Interscience Publishers, New York, 1963, page 203.

for the target nuclei. We also accept this approximation which is good for incident energies large compared to target temperature and which places our theory on a par with Longmire.

II. Theory -- Kinematics. The average energy loss per unit path from elastic scattering, $d\bar{W}_s/dx$, is given by the average energy loss from scattering per collision, $\overline{\Delta W}_s$, times the probability of a collision per scattering center, σ_T, times the number (atoms) of scattering centers per unit volume, N_A. Thus,

$$\frac{\overline{dW}_s}{dx} = \overline{\Delta W}_s \cdot \sigma_T \cdot N_A. \quad (1)$$

The average energy loss per collision, $\overline{\Delta W}_s$, is related to the energy loss from an elastic scattering, $\Delta W_s(\theta)$, at center of mass angle, θ, by the expression

$$\overline{\Delta W}_s = \frac{1}{\sigma_T}\int_{\phi=0}^{2\pi}\int_{\cos\,\theta=-1}^{1} \Delta W_s(\theta)\cdot\sigma_s'(\theta)\cdot d\Omega, \quad (2)$$

$d\sigma_s/d\Omega \equiv \sigma_s'(\theta)$ being the differential elastic scattering cross section at angle θ into solid angle $d\Omega$. The total cross section is σ_T.

Suppose a particle of mass m_1 and charge Z_1 with velocity v_o strikes a stationary particle of mass m_2 and charge Z_2 so as to be scattered through an angle θ_L in center-of-mass angle θ with a probability σ_s', the recoil particle having a laboratory velocity v_L, and giving rise to a center-of-mass recoil velocity of $v_2 \equiv V$, the velocity of the center of mass, (See Fig. 1.).

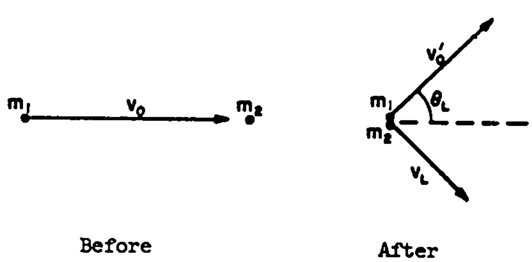

Fig. 1. Laboratory system.

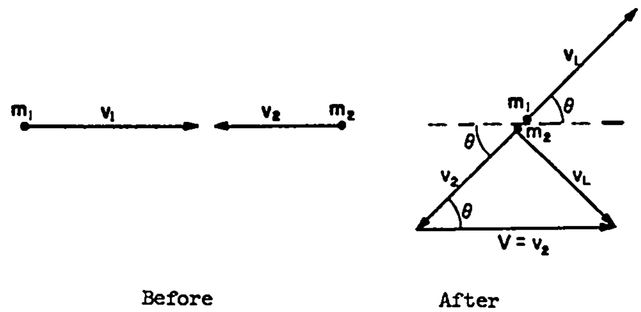

Fig. 2. Center-of-Mass system.

In the center-of-mass triangle, $v_2\ v_2\ v_L$, of Fig. 2 "after," the law of cosines gives

$$v_L^2 = v_2^2 + v_2^2 - 2v_2^2\cos\theta = 2v_2^2(1-\cos\theta). \quad (3)$$

Now the energy loss per scattering, ΔW_s, is just the energy gained by m_2,

$$\Delta W_s = \frac{1}{2}m_2 v_L^2 = m_2 v_2^2(1-\cos\theta) = \frac{m_2 m_1^2}{(m_1+m_2)^2}v_o^2(1-\cos\theta), \quad (4)$$

where we have used the constancy of momentum,

$$m_1 v_o = (m_1 + m_2)\ V = (m_1 + m_2)\ v_2.$$

Substitution of (4) with $W \equiv \frac{1}{2}m_1 v_o^2$, $d\Omega = 2\pi d\cos\theta$, and $Z \equiv \cos\theta$, yields the logarithmic energy loss from elastic scattering,

$$\frac{10^{24}}{N_A W}\cdot\frac{\overline{dW}_s}{dx} = \frac{4\pi\ m_1 m_2}{(m_1+m_2)^2}\int_{-1}^{1}\sigma_s'(Z)\cdot(1-Z)dZ, \quad (5)$$

where σ_s' is in barns/steradian, N_A is in atoms/cubic centimeter, and x is in centimeters.

III. Theory -- Unlike Particles. Heretofore our theory applied to like or unlike particles. We now specialize to the form seen by unlike particles only.

The differential scattering amplitude, f, is related to σ_s' by [2]

[2] Any text on scattering theory; for example, A. Messiah, Quantum Mechanics, John Wiley, New York, 1958, page I372.

$$\sigma_s' = |f|^2. \quad (6)$$

For unlike particles under both Coulomb and nuclear forces, f is of the form

$$f = f_c + f_N, \quad (7)$$

with

$$f_c = -\frac{\gamma}{2k \sin^2 \frac{\theta}{2}} \exp\left[-i\gamma \ln(\sin^2 \frac{\theta}{2}) + 2i\sigma_0\right],$$

$$\gamma \equiv \frac{Z_1 Z_2 e^2}{\hbar v}; \quad \sigma_0 \equiv \arg \Gamma(1 + i\gamma), \quad (7a)$$

and

$$f_N = \sum_{\ell=0} a_\ell P_\ell(\cos\theta), \quad (7b)$$

with a_ℓ being constants and P_ℓ being the Legendre polynomials.

Equation (7a) is from Messiah, p. I421, 422; (7b) is from p. I386. Substituting

$$2 \sin^2 \frac{\theta}{2} = 1 - \cos\theta = 1 - Z$$

and using the fact that $P_\ell(Z)$ is just a polynomial of degree ℓ, we have

$$f_c = -\frac{\gamma}{k(1-Z)} \exp\left[- i\gamma \ln\left(\frac{1-Z}{2}\right) + 2i\sigma_0\right] \quad (8a)$$

and

$$f_N = \sum_{\ell=0}^{\bar{\ell}} b_\ell Z^\ell, \quad (8b)$$

where b_ℓ are (new) constants.

Observing that f_c has a pole for Z = 1, we expect the Coulomb term to dominate near such a pole and, as it turns out, it is then also small for Z << 1 (large angles). Further, the interference term is important only for a small range of Z; moreover, the exponent of (8a) is only weakly dependent on Z through the logarithm. In addition, the pure Coulomb term $|f_c|^2$, which is just classical Rutherford scattering, does not depend on the exponent at all.

Consequently we can make a good* approximation by writing

$$f_c \cong \frac{b_{-1}}{(1 - Z)}, \quad (9)$$

where b_{-1} is taken constant with Z (which is done for the interfence term only).

Actually we can describe the effect of b_{-1} variation to any desired degree of accuracy by simply choosing $\bar{\ell}$ sufficiently large. Equations (7), (8b), and (9) now yield the form of our scattering amplitude:

$$f = \frac{b_{-1}}{(1-Z)} + b_0 + b_1 Z + \ldots + b_{\bar{\ell}} Z^{\bar{\ell}}. \quad (10)$$

Insertion into (6) gives the differential scattering cross section with new (real) constants, d:

$$\sigma_s' = \sigma_R' + \frac{d_{-1}}{(1-Z)} + d_0 + d_1 Z + \ldots + d_{2\bar{\ell}} Z^{2\bar{\ell}}, \quad (11)$$

where the well-known Rutherford scattering cross section is

$$\sigma_R' \equiv \frac{|b_{-1}|^2}{(1-Z)^2} = \frac{\gamma^2}{4k^2 \sin^4 \frac{\theta}{2}} = \frac{Z_1^2 Z_2^2 e^4 (m_1 + m_2)^2}{4W^2 m_2^2 (1-Z)^2}. \quad (12)$$

The remainder of σ_s' we label

$$\sigma_{IN}' \equiv \sigma_s' - \sigma_R', \quad (13)$$

the interference term $\frac{d_{-1}}{(1-Z)} \equiv \sigma_I'$, and the nuclear term

$$\sigma_N' = \sum_{\ell=0}^{2\bar{\ell}} d_\ell Z^\ell,$$

although the labels thus made are not strictly accurate (the nuclear part contains some of the interference contributions). Since we desire the elastic scattering energy loss less the pure Coulomb part, we substitute (11) less σ_R' into (5) and perform the integration, obtaining thereby the logarithmic energy loss from nuclear scattering and the interference of that with nuclear coulomb scattering:

---

* In D(p,p)D, for example, we calculate that for $\bar{\ell}$ = 2, E = 1.5 MeV the variation of b_{-1} has an effect of less than 2.4% on the full result, in the range from Z = 0.9 to Z = 0.5 (the region of greatest interference). There is less error for higher $\bar{\ell}$, greater for lower E. See "use" discussion after (14).

$$\Psi = \frac{10^{24}}{N_A\, W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} =$$

$$\frac{8\pi\, m_1 m_2}{(m_1 + m_2)^2}\left[d_{-1} + \sum_{\substack{n=0 \\ \text{even}}}^{2\bar{\ell}} \frac{d_n}{n+1} - \sum_{\substack{n=1 \\ \text{odd}}}^{2\bar{\ell}-1} \frac{d_n}{n+2}\right] \qquad (14)$$

which is our result for the average energy loss of a particle traversing an unlike material of atom density, N_A, whose total differential scattering cross section in barns per steradian is of the form (11) and (12). Call this quantity Ψ.

Our most accurate evaluation of the energy loss, Ψ, is by digital computer in that we effectively fit (to any desired accuracy) the experimental data by the form (11) using a least-squares procedure. The coefficients thereby obtained are inserted into a form of (14) to yield Ψ.

For completeness, however, we exhibit a method of hand calculation. We begin with $\bar{\ell} = 1$ so that

$$\sigma_s' = \sigma_R' + \frac{d_{-1}}{(1-Z)} + d_0 + d_1 Z + d_2 Z^2 , \qquad (15)$$

and the bracket of (14) is

$$\left[d_{-1} + d_0 - \frac{d_1}{3} + \frac{d_2}{3}\right]. \qquad (16)$$

σ_R' can be calculated and σ_N' extrapolated to give a crude estimate of d_{-1} from

$$\sigma_I' \equiv \sigma_s' - \sigma_R' - \sigma_N' = \frac{d_{-1}}{(1-Z)} .$$

Then, labelling $\sigma_\pm' \equiv \sigma_N'$ $(Z = \pm 1)$, $\sigma_0' \equiv \sigma_N'$ $(Z = 0)$, we find that

$$\sigma_+' = d_0 + d_1 + d_2, \quad \sigma_-' = d_0 - d_1 + d_2, \quad \sigma_0' = d_0. \qquad (17)$$

Solving for the d's and substituting in (16) and (14), we have

$$\Psi = \frac{10^{24}}{N_A\, W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} = \frac{8\pi\, m_1 m_2}{3(m_1 + m_2)^2}\left[3d_{-1} + 2\sigma_0' + \sigma_-'\right]. \qquad (18)$$

The prescription for use of this formula is to calculate σ_R', (12) extrapolate σ_N' from low Z (or high θ) in the center of mass, in order to obtain σ_I' and thereby d_{-1}, and finally use the measured σ_s' at Z = 0, -1 to get σ_0' and σ_-', $\sigma_N' = \sigma_s' - \sigma_R' - \sigma_I'$. Note that σ_+', which is very difficult to estimate, does not appear in (18) because forward scattering implies zero energy loss.

Similarly, for more accuracy, one may put $\bar{\ell} = 2$ and then one needs to add two more estimates of σ_N', say, at $Z = \pm\frac{1}{2}$ -- call them $\sigma_{\pm\frac{1}{2}}$, respectively.

$$\Psi = \frac{10^{24}}{N_A\, W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} =$$

$$\frac{8\pi\, m_1 m_2}{45(m_1 + m_2)^2}\left[45d_{-1} + 8\sigma_{+\frac{1}{2}}' + 6\sigma_0' + 24\sigma_{-\frac{1}{2}}' + 7\sigma_-'\right], \qquad (19)$$

where again σ_+' drops out.

Such hand calculations can give only crude answers since they depend critically on the accuracy of the selected cross sections at Z = 0, ±1/2, -1. Use of a digital computer allows us to fit the whole range of the cross sections with a least-squares polynomial curve of any desired degree. Thus we can determine the parameters in the energy deposition as accurately as the experiment itself permits.

Accordingly, we return to (11) and form the expression

$$\sum \equiv (1 - Z)\left[\sigma_s' - \sigma_R'\right] = \sum_{n=0}^{2\bar{\ell}+1} e_n Z^n . \qquad (20)$$

The multiplication of the polynomial $\Sigma\, d_n Z_n$ of (11) by $(1 - Z)$ leads to the polynomial Σ of (20) with the constants

$$e_0 = d_{-1} + d_0, \quad e_{\substack{n\neq 0 \\ n\neq 2\bar{\ell}+1}} = d_n - d_{n-1}, \quad e_{2\bar{\ell}+1} = -d_{2\bar{\ell}} .$$

Solving for the d's and substituting into (14) gives

$$\Psi = \frac{10^{24}}{N_A\ W} \cdot \frac{d\bar{W}}{dx} = \frac{8\pi\ m_1 m_2}{(m_1 + m_2)^2} \cdot \sum_{m=0}^{\bar{\ell}} \frac{e_{2m}}{2m+1} . \tag{21}$$

Thus, for each given energy, W, the computer is given an $\bar{\ell}$ and a table of σ_s' versus Z. From these values, the Σ of (20) is determined for each Z and a standard least-squares polynomial curve of degree $2\bar{\ell} + 1$ is formed[3] yielding the coefficients e_n. The even indexed coefficients are then used in (21) to evaluate the desired energy loss, Ψ.

The derived least-square values of σ_s', as well as the sum of the variances of Σ, are also printed by the computer so that the accuracy of the fit can be determined. In every case of our examples, the fits were within experimental accuracy.

IV. Theory -- Like Particles. As is well known, identical incident and target particles lead to some theoretical complication and also to an advantageous symmetry. Even in the Coulomb term, these effects change the whole character of the cross section, and, further, the Coulomb cross sections differ for different spin particles. The particles we are interested in have spins of 0, $\frac{1}{2}$, 1, so we shall confine ourselves to these values. Extension to other spins, we believe, is straightforward. Messiah,[2] pages 606 to 608, may be referred to for the underlying theory. Note that all cross sections are symmetric about 90° in the center of mass, so our expansions are even in Z.

A. Spin 0. Since we cannot distinguish whether the incident or target particle is detected at angle $\theta\phi$, the quantum mechanical cross section for observing either particle at $\theta\phi$ becomes

$$\sigma_{so}'(\theta\phi) = |f(\theta\phi) \pm f(\pi - \theta,\ \phi + \pi)|^2 \text{ spinless} \tag{22}$$

+ for a boson, - for a fermion, if $f(\theta\phi)$ is the distinguishable scattering amplitude. Thus for zero spin, bosons (for example α on α), using the form (8) and (10), $(\cos(\pi - \theta) = -\cos\theta = -Z)$:

$$f(\theta) + f(\pi-\theta) = f(Z) + f(-Z) = f_c(Z) + f_c(-Z) + 2 \sum_{\substack{n=0 \\ \text{even}}}^{\bar{\ell}} b_n Z^n . \tag{23}$$

Taking the square, the Coulomb part, $\sigma_{co}' = |f_c(Z) + f_c(-Z)|^2$, is composed of three terms:

$$|f_c(Z)|^2 = \frac{\gamma^2}{k^2(1-Z)^2} ,$$

which is just σ_R' of (12), incident scattering in center of mass angle θ.

$$|f_c(-Z)|^2 = \frac{\gamma^2}{k^2(1+Z)^2} ,$$

which is σ_R' except that the struck nucleus goes off in the direction θ.

$$f_c(Z)\ f_c^*(-Z) + f_c^*(Z)\ f_c(-Z) = \frac{2\gamma^2 \cos\left(\gamma\ \ell n\ \frac{1-Z}{1+Z}\right)}{k^2(1-Z^2)} \tag{24}$$

(after some manipulation) which is the Coulomb-Coulomb interference term, a wholly nonclassical effect.

Thus the Coulomb part has the form* (using in (12) $Z_1 = Z_2$ and $m_1 = m_2$)

$$\sigma_{co}' = \frac{Z_2^4 e^4}{W^2} \left[\frac{1}{(1-Z)^2} + \frac{1}{(1+Z)^2} + \frac{2\cos\left(\frac{Z_2^2 e^2}{\hbar\ v_o}\ \ell n\ \frac{1-Z}{1+Z}\right)}{(1-Z^2)} \right], \tag{25}$$

with

$$\sigma_{so}' = \sigma_{co}' + \sigma_{NI}' , \tag{26}$$

and (after further manipulation) the remainder of (23) squared is of the form

$$\sigma_{NI}' = \frac{d_-}{(1-Z^2)} + \sum_{m=0}^{\bar{\ell}} d_{2m} Z^{2m} , \tag{27}$$

[3]See D. D. McCracken and W. S. Dorn, Numerical Methods and Fortran Programming, John Wiley, New York, 1964, pages 262 - 275.

*Note that these Coulomb terms differ for different spins alone; thus, this spin zero term is not the same as that for spin 1/2 particles of the same mass and charge, the latter called Mott scattering, nor is it the same as spin 1, etc.

which is the nuclear-Coulomb interference term plus the pure nuclear terms. These are not form dependent on spin, so we need not carry explicit spin dependence in them.

It appears most useful to follow the most energetic resulting particle from each collision, so we integrate on center of mass angle from 0 to 90^o only (Z from 0 to 1)*, in substituting (27) into (5) to obtain $(m_1 = m_2)$

$$\Psi \equiv \frac{10^{24}}{N_A W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} =$$

$$\pi d_- \ell n\, 2 + \pi \sum_{m=0}^{\mathcal{L}} \frac{d_{2m}}{(2m+1)(2m+2)}, \qquad (28)$$

which is our energy deposition per unit path length of the most energetic spin zero particle from hadronic and hadronic-Coulomb interference elastic scattering, when the total elastic scattering has the form (26), (25), and (27).

For hand calculation we again define $\sigma'_{\pm,\pm\frac{1}{2},0}$ to be

$$\sigma'_N \equiv \sigma'_s - \sigma'_{co} - \sigma'_I \ (Z = \pm 1, \pm\tfrac{1}{2}, 0, \text{ respectively});$$

we calculate σ'_{co} from (25), estimate first σ'_N from σ'_s less σ'_{co}, and then estimate σ'_I to obtain

$$d_- \text{ in } \sigma'_I \approx \frac{d_-}{(1-Z^2)}.$$

Algebraic manipulation then yields, for $\mathcal{L} = 1$

$$\Psi = \frac{10^{24}}{N_A W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} =$$

$$\frac{\pi}{12}\left[(12\ \ell n\ 2)d_- + 5\sigma'_0 + \sigma'_+\right], \qquad (29)$$

or

$$\frac{\pi}{6}\left[(6\ \ell n\ 2)d_- + \sigma'_0 + 2\sigma'_{\frac{1}{2}}\right],$$

*We also avoid thereby counting particles twice.

which formula is exact when

$$\sigma'_s = \sigma'_{co} + \frac{d_-}{(1-Z^2)} + d_o + d_2 Z^2.$$

For $\mathcal{L} = 2$,

$$\Psi = \frac{\pi}{60}\left[60 d_-\ \ell n\ 2 + 13\sigma'_0 + 16\sigma'_{\frac{1}{2}} + \sigma'_+\right], \qquad (30)$$

exact for $\sigma'_s = \sigma'_{co} + \dfrac{d_-}{(1-Z^2)} + d_0 + d_2 Z^2 + d_4 Z^4$.

For machine calculation, we form

$$\Phi \equiv (1 - Z^2)(\sigma'_{so} - \sigma'_{co}) = \sum_{m=0}^{\mathcal{L}+1} e_{2m} Z^{2m} \qquad (31)$$

since multiplication of (27) by $(1-Z^2)$ yields an even polynomial of degree $2(\mathcal{L}+1)$, with $e_o = d_- + d_0$, $e_{2n} = d_{2n} - d_{2n-2}$ for n = 1 to $\mathcal{L}$, and $e_{2\mathcal{L}+2} = -d_{2\mathcal{L}}$.

Solving for the d's and substituting in (28) gives

$$\Psi = \frac{10^{24}}{N_A W} \cdot \left.\frac{\overline{dW_s}}{dx}\right|_{NI} =$$

$$\pi\ \ell n\ 2 \sum_{n=0}^{\mathcal{L}+1} e_{2n} - \pi \sum_{n=0}^{\mathcal{L}} e_{2n+2}\left[\sum_{m=0}^{n} \frac{1}{(2m+1)(2m+2)}\right]. (32)$$

As in the unlike particle case, for each given energy, W, the computer is presented a value of $\mathcal{L}$ and a table of σ'_s vs Z. For each value of Z, a value of σ_{co} is calculated and then Φ is formed from (31). A least-squares polynomial of even powers and of degree $2\ell + 2$ is fitted to the Φ vs Z values. The resulting coefficients e_m are used in (32) to obtain the energy deposition, Ψ. The computer also prints the sum of the variances plus the derived σ'_s vs Z in order to check the goodness of fit.

B. Spin $\frac{1}{2}$. Spin $\frac{1}{2}$ particles are fermions so that the total wave function must be antisymmetric. Thus, when the total spin, S, of the incident and target particle is symmetric, S = 1 (triplet state-probability $\frac{3}{4}$), the space part must be antisymmetric. When the spin part is antisymmetric, S = 0 (singlet-probability $\frac{1}{4}$), the space part is symmetric. Thus the scattering cross section has the form (symmetric in ϕ, i.e., no polarization measurements, etc.)

$$\sigma'_{s\frac{1}{2}} = \frac{3}{4}\left|f_t(\theta) - f_t(\pi-\theta)\right|^2 + \frac{1}{4}\left|f_s(\theta) + f_s(\pi-\theta)\right|^2, \quad (33)$$

with f_t, f_s being the triplet and singlet scattering amplitudes, respectively.

Similarly to (23), the two terms above have the forms

$$f_t(\theta) - f_t(\pi-\theta) = f_c(Z) - f_c(-Z) + 2 \sum_{\substack{n=1 \\ \text{odd}}}^{\bar{\ell}} a_n Z^n \quad (34)$$

and

$$f_s(\theta) - f_s(\pi-\theta) = f_c(Z) + f_c(-Z) + 2 \sum_{\substack{n=0 \\ \text{even}}}^{\bar{\ell}} a_n Z^n. \quad (35)$$

Squaring, the specifically Coulomb terms are easily obtained and differ only in the numerical coefficient of the Coulomb-Coulomb interference, (24).

$$\sigma'_{c\frac{1}{2}} = \frac{Z_2^4 e^4}{W^2}\left[\frac{1}{(1-Z)^2} + \frac{1}{(1+Z)^2} - \frac{\cos\left(\frac{Z_2^2 e^2}{\hbar\, v_0} \ell n\, \frac{1-Z}{1+Z}\right)}{(1-Z^2)}\right], \quad (36)$$

which is the well-known Mott formula for the Coulomb scattering of spin $\frac{1}{2}$ identical particles.

The remaining terms of (33), σ'_{NI}, are, in fact, of the form (27)

$$\sigma'_{NI} = \frac{d_-}{(1-Z^2)} + \sum_{m=0}^{\bar{\ell}} d_{2m} Z^{2m} \quad (27)$$

for some constants d. This can be shown by a tedious calculation, but is more simply noted by observing that the form of (34) is

$$\frac{2Z\, a_-}{(1-Z^2)} + 2a_1 Z + 2a_3 Z^3 + \ldots ,$$

and that of (35) is

$$\frac{2a_-}{(1-Z^2)} + 2a_0 + 2a_2 Z^2 \ldots ,$$

which upon squaring and incorporation of a_-^2 terms into (36) leaves us with terms exactly of the form (27) and of maximum degree $2\bar{\ell}$.

Since σ'_{NI} is the same in form for spin $\frac{1}{2}$ as for spin 0, the same formalism applies, and, except for the pure Coulomb cross section which should now be $\sigma'_{c\frac{1}{2}}$ of (36), the hand calculations of Ψ are given by (29), $\bar{\ell} = 1$, and (30), $\bar{\ell} = 2$. The machine calculation is given by (32).

C. Spin 1. Spin 1 particles are bosons with total symmetric wave functions; therefore, the system of incident and target identical particles must be space symmetric for total spin, S = 2, probability $\frac{5}{9}$; space antisymmetric for S = 1, probability $\frac{3}{9}$; and space symmetric for S = 0, probability $\frac{1}{9}$. Consequently (ϕ symmetric),

$$\sigma'_{S1} = \frac{5}{9}\left|f_2(\theta) + f_2(\pi-\theta)\right|^2 + \frac{3}{9}\left|f_1(\theta) - f_1(\pi-\theta)\right|^2 + $$

$$+ \frac{1}{9}\left|f_0(\theta) + f_0(\pi-\theta)\right|^2. \quad (37)$$

Again, although differing in value for different hadronic amplitudes, f_i, indeed, differing among the several spin states, S = 2, 1, 0 as well as from spin $\frac{1}{2}$ of the preceding section, the forms of the amplitudes are

$$f_1(\theta) - f_1(\pi-\theta) = f_c(Z) - f_c(-Z) + 2 \sum_{\substack{n=1 \\ \text{odd}}}^{\bar{\ell}} a_{1n} Z^n \quad (34A)$$

and

$$f_i(\theta) + f_i(\pi-\theta) = f_c(Z) + f_c(-Z) + 2 \sum_{\substack{n=0 \\ \text{even}}}^{\bar{\ell}} a_{in} Z^n. \quad (35A)$$

Again, squaring leads to the pure Coulomb term differing only in the interference part, (24),

$$\sigma'_{c1} = \frac{Z_2^4 e^4}{W^2}\left[\frac{1}{(1-Z)^2} + \frac{1}{(1+Z)^2} + \frac{2}{3}\, \frac{\cos\left(\frac{Z_2^2 e^2}{\hbar\, v_0} \ell n\, \frac{1-Z}{1+Z}\right)}{(1-Z^2)}\right], \quad (38)$$

which differs from both spin 0. Eq. (25), and spin $\frac{1}{2}$, the Mott formula, Eq. (36).

The remainder of σ'_{s1}, namely the absolute squares of (34A) and (35A) less pure Coulomb, leads again to terms identical to the form of (27) so the spin 0 formalism applies except that all pure Coulomb terms should have σ'_{c1}, Eq. (38), as the cross section rather than σ'_{c0}.

The hand calculations of ψ are given by (29), $\bar{\ell} = 1$, and (30), $\bar{\ell} = 2$. The machine calculation is given by (32). But in fitting the experimental cross sections, the pure Coulomb part is, of course, σ'_{c1}.

V. Numerical Examples.

A. Energy deposition by elastic scattering of protons in deuterium. The calculations in this section are based on the comprehensive summary by Seagrave.[4] We used the neutron cross sections of Allen et al.[5] to estimate the energy loss at 0.1- and 0.2-MeV incident protons. Formula (20) was used to fit the data as described, with data-extrapolated end points added. That is, the machine fit can go wild beyond the range of experimental points, especially when the data do not cover small enough or large enough angles. In these instances a graphical or linear extrapolated point was added to preserve the expected form. We used the minimum-degree polynomial consistent with a fit as good as experiment allowed. $\bar{\ell}$ is, of course, expected to increase at higher energy, and so it was found. We used $\bar{\ell} = 1$ at 0.1 and 0.2 MeV and ranged to $\bar{\ell} = 4$ at 14 Mev, depending on the data.

The results are shown in Fig. 3, which gives the energy loss from nuclear forces elastic scattering plus the interference of that scattering with nuclear Coulomb scattering as given by (21). The values do not include nuclear Coulomb scattering. The points are keyed to the accompanying experimental references.

B. Energy deposition by elastic scattering of alphas in deuterium. The angular $He^4(d,d)He^4$ cross-section data upon which this section is based were frequently more sporadic than, for example, those in the preceding D(p,p)D. Indeed, the points labelled Blair et al. and Galonsky et al. in the references for Fig. 4 are a combination of their data, with Blair et al. providing the small-angle data and Galonsky et al. giving the mid and large angles. Since our polynomial fit is to the experimentally derived points of Σ, Eq. (20), the resulting curve of σ'_s vs Z can go wild between widely spaced experimental points. We controlled these excursions in occasional sparse data regions by adding a point obtained from smooth graphs of the data. The resulting fits are thereby optimized and appear to be as good as experiment. However, in some instances, particularly near the resonance for $E_\alpha = 2.13$ MeV, experiment and, consequently, the fits were obviously erratic. The results from such fits were given less weight in the final curve.

D. C. Dodder and his collaborators[6] kindly provided us with cross sections based on their phase analysis of the data. The results from such data enabled us to detail the $E_\alpha = 2.13$-MeV resonance better and were crucial in determining the curve at low energy.

As before, we fit the data using (20) and thereby determined the parameters e that were used to calculate the energy deposition of alpha particles per path length in deuterium by means of (21), exhibited in Fig. 4. We used high $\bar{\ell}$ if the data warranted; actual $\bar{\ell}$'s used ranged from 2 to 5. Again, the results are only for nuclear scattering and that interfering with Coulomb. The points are keyed to the accompanying references.

C. Energy deposition by elastic scattering of deuterons in deuterium. The D(d,d)D data as a function of angle ranged from 4 points to 25 points, but fortunately the angular momentum involved was low, $\ell \leq 2$, implying $\bar{\ell} = 2$ except for possible added flexibility needed for interference term variation, see (9) and following remark. Indeed, we often got very good fits even at high energy with $\bar{\ell} = 2$, although wherever it appeared more advantageous we went to $\bar{\ell} = 3$ or even 4. The only added point was to $E_d = 0.87$ MeV, which resulting energy loss callation was rejected, see Fig. 5.

Dodder et al.[6] provided theoretical cross sections. These were especially valuable at low energy

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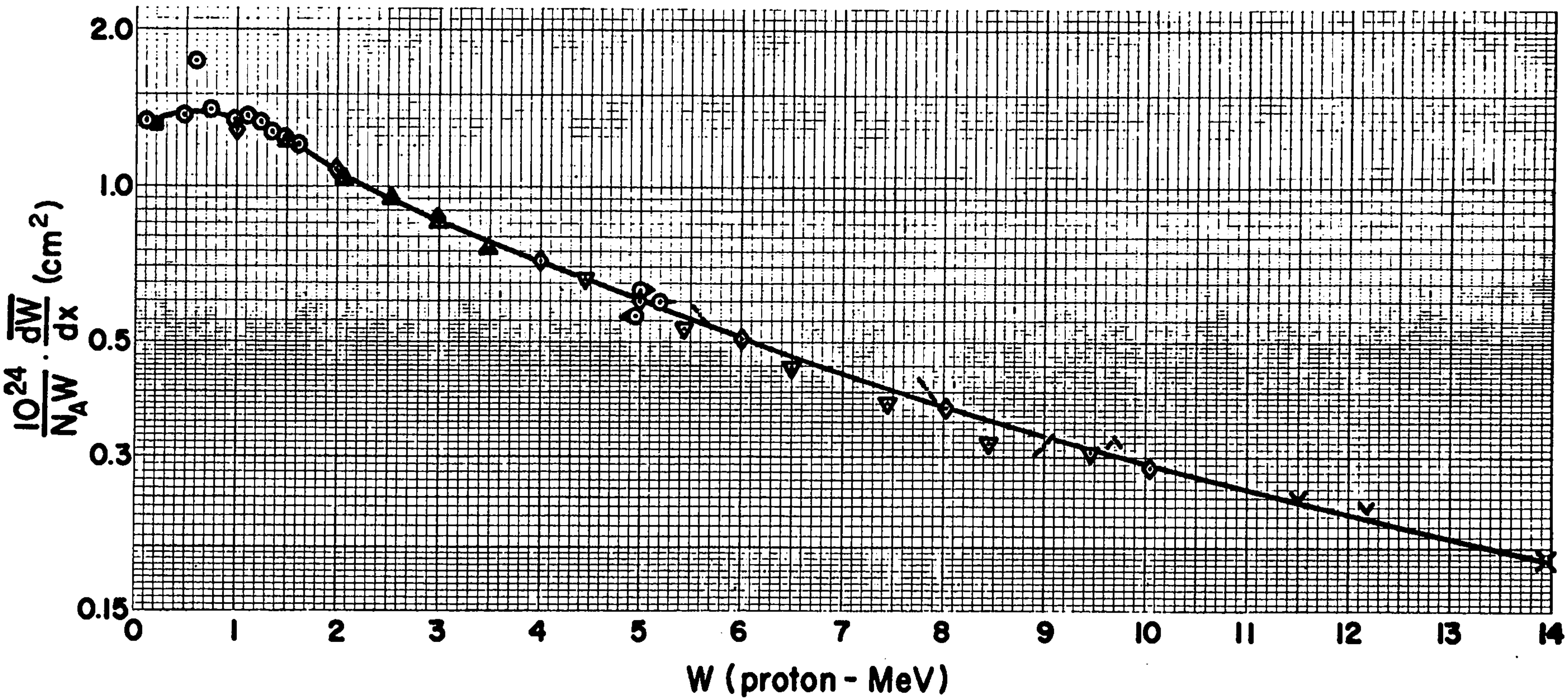

Fig. 3. Energy gain of D from D(p,p)D (includes Coulomb interference but not Coulomb).

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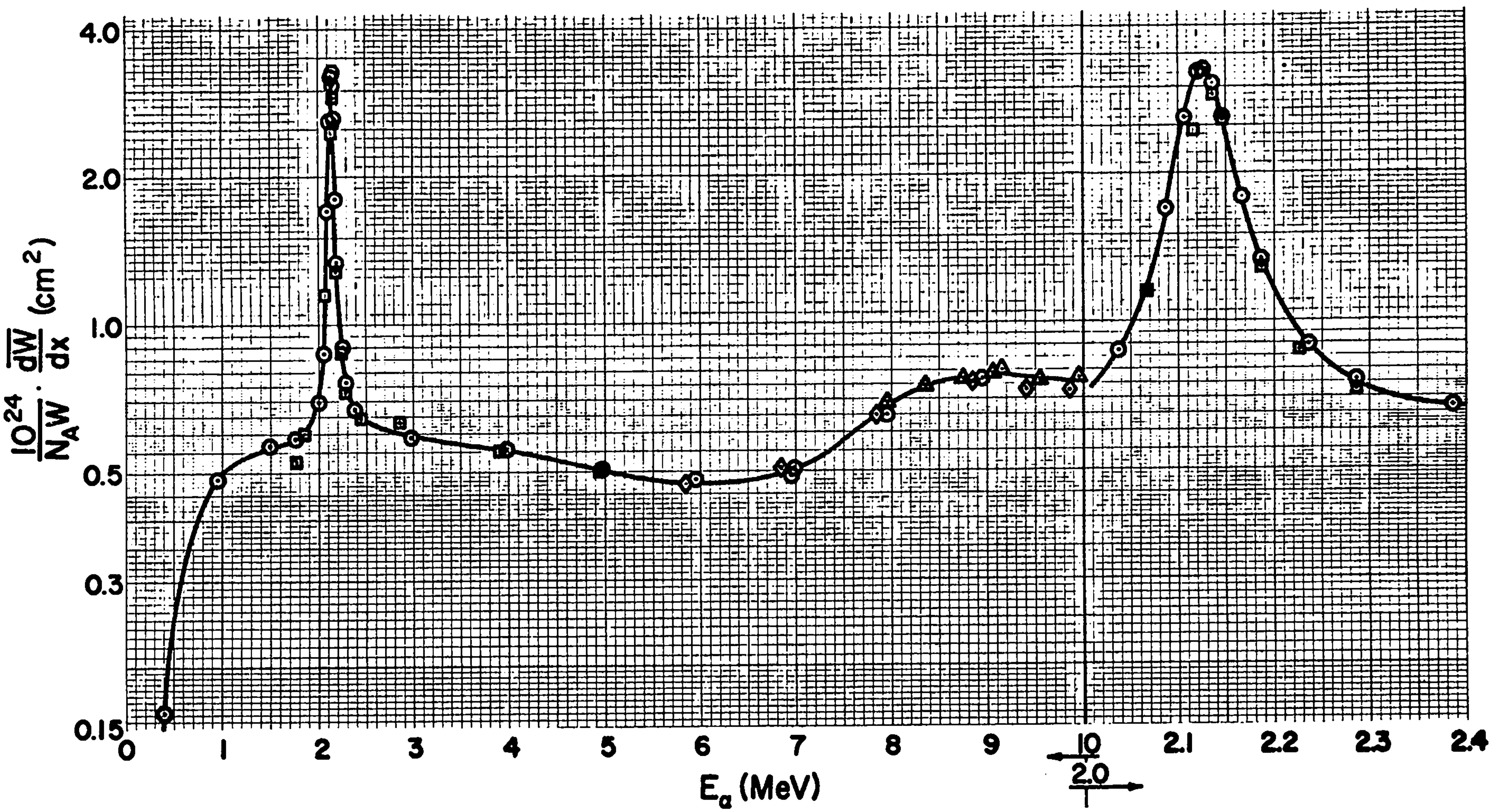

Fig. 4. Energy gain of D from D(α,α)D (includes Coulomb interference, but <u>not</u> Coulomb).

<u>References for Fig. 4.</u> All are σ′ $He^4(d,d)He^4$.

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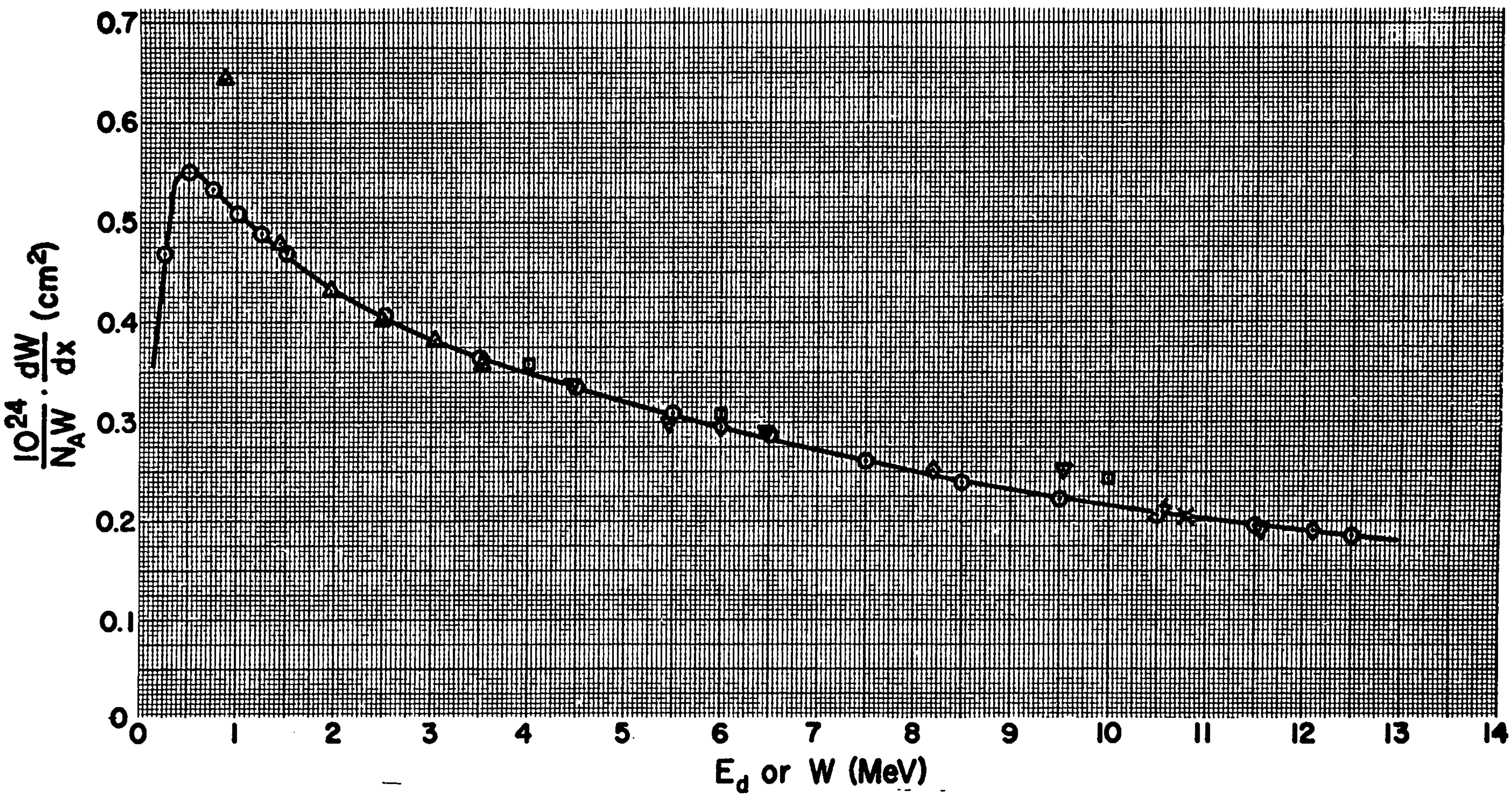

Fig 5. Energy gain from D(d,d)D (includes Coulomb interface, but <u>not</u> Coulomb) most energetic D followed, i.e., θ cm from 0° to 90° only.

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for they enabled us to extend our determination of energy loss below 1 MeV.

In this identical particle, spin 1, case (31) was used to fit the data, but with (38) as the pure Coulomb part. The resulting parameters e were substituted into (32) to obtain the energy loss from nuclear forces elastic scattering plus nuclear-Coulomb interference per path length of the most energetic deuteron. Note that nuclear Coulomb scattering is not included. These results are plotted in Fig. 5, the points being keyed to the accompanying references.

VI. Index and Summary of Results. This paper gives energy loss or deposition due to nuclear forces elastic scattering plus the nuclear force-nuclear Coulomb interference, but excluding the pure nuclear Coulomb scattering. The pure nuclear Coulomb scattering cross-section formulas, however, are given. The latter may be found as follows.

For unlike particles, Eq. (12), p. 3.
For like particles of spin 0, Eq. (25), p. 5.
For like particles of spin $\frac{1}{2}$, Eq. (36), p. 7.
For like particles of spin 1, Eq. (38), p. 7.

A. Energy loss -- theory

1. Unlike particles. For a cross section, σ'_s, described by (11) and (12), page 3, the nuclear and interference energy loss is given by (14). To calculate the loss from a graph of σ'_s vs Z by hand, we define $\sigma'_{0,\pm\frac{1}{2},\pm} \equiv \sigma'_N$ ($Z \equiv \cos\theta$, $\pm\frac{1}{2}$, ±1, respectively) as being the nuclear part and d_{-1} as the interference constant, see (15), page 4; the energy losses are then given for ℓ = 1 and 2 by (18) and (19), page 4, respectively. Our method for machine calculation is to fit the experimental cross-section data minus pure Coulomb in the form of (20), page 4, with a least-squares polynomial curve, thus evaluating the coefficients e_{2m} used to determine Ψ in (21), page 5.

2. Like particles. Except for the spin-dependent Coulomb formulas noted above, (25), (36), and (38), the nuclear and interference parts have the same form among like particles, and so can be described together. With that qualification, then, for a cross section, σ'_s, described by (27), page 5, plus the appropriate pure Coulomb part above, (25), (36), or (38), the nuclear and interference energy loss is given by (28). To pick off the energy loss from a graph by hand, one estimates the nuclear part, σ'_0, $\sigma_{+\frac{1}{2}}$, σ'_+, and the interference term using the appropriate σ'_c and the form (27). Then (29) or (30), page 6, yields the energy loss. The method for machine calculation is to fit the experimental cross section less Coulomb in the form of (31), page 6, with a least-squares polynomial of even powers whose coefficients, e_{2n}, enable the result, Ψ, of (32), page 6, to be evaluated.

B. Energy loss -- numerical values.

1. For D(p,p)D the results are given in Fig. 3 page 9.
2. For D(α,α)D -- Fig. 4, page 10.
3. For D(d,d)D -- Fig. 5, page 11.

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