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Solution of a Nonlinear Integrodifferential Equation by the Method of Upper and Lower Functions

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Solution of a Nonlinear Integrodifferential Equation by the Method of Upper and Lower Functions

by

P. K. Zeragiia

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SOLUTION OF A NONLINEAR INTEGRODIFFERENTIAL EQUATION BY THE METHOD OF UPPER AND LOWER FUNCTIONS

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P. K. Zeragiia

ABSTRACT

The nonlinear integrodifferential equation

$$y''(x) = F\left\{x, \lambda \int_a^x K[x,t,y(t),y'(t)]dt\right\} \qquad (1)$$

is considered. The problem is to find a function, y(x), that has continuous first- and second-order derivatives in the closed interval [a,b], and in this interval satisfies Eq. (1) and the boundary conditions $y(a) = y_0$ and $y'(a) = y_0'$, where a, b, y_0, and y_0' are given numbers. With the aid of some functional inequalities, sequences of upper and lower functions are constructed for certain conditions of the functions F and K. It is shown that these sequences converge to the solution of Eq. (1) for any given finite interval [a,b].

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1. Let us consider a nonlinear integrodifferential equation of the form

$$y''(x) = F\left\{x, \lambda \int_a^x K[x,t,y(t),y'(t)]dt\right\} \quad . \qquad (1)$$

Let us set up the problem of finding the function, y(x), that is continuous, that has continuous first- and second-order derivatives in the interval [a,b], and that in this range satisfies Eq. (1) and the initial conditions

$$y(a) = y_0 \text{ and } y'(a) = y_0' \quad , \qquad (2)$$

where a, b, y_0, and y_0' are given numbers.

Using the method of contractive mappings, Zhenkhen[1] proved the existence of a unique solution of Eq. (1) with a fixed upper limit of an integral and for a sufficiently small $|\lambda|(b - a)$. In this study, with the help of functional inequalities, we set up, for definite conditions of the functions F and K, sequences of upper and lower functions that converge to the desired solution of Eq. (1) in any finite interval [a,b].

We assume that the right-hand part of Eq. (1) fulfills the following conditions.

a. The function F(x,u) is continuous and limited together with the derivative F_u and satisfies the condition $\lambda F_u > 0$ in the region $D = \{a \leqq x \leqq b, -\infty < u < \infty\}$.

b. The function $K(x,t,y,y')$ is continuous and limited together with the derivatives K_y and $K_{y'}$ that satisfy the conditions $K_y > 0$ and $K_{y'} \geqq 0$ in the region

$$B = \{a \leqq t \leqq x \leqq b, -\infty < y, y' < \infty\} \quad .$$

2. Given the above conditions, the following propositions are valid.

Theorem 1. If the function z(x) is continuous and has continuous first- and second-order derivatives in the interval $[a,b]$, it satisfies the initial conditions of Eq. (2) and the inequality

$$z''(x) > F\left\{x, \lambda \int_a^x K[x,t,z(t),z'(t)]dt\right\} \quad ,$$

$$(a \leq x \leq b) \quad . \qquad (3)$$

Then

$$z(x) > y(x) \quad \text{npu } x \epsilon (a,b) \quad , \qquad (4)$$

where $y(x)$ is the continuous function having continuous first- and second-order derivatives in the interval $[a,b]$, and satisfying Eq. (1) and the initial conditions of Eq. (2).

Proof. We will get $z''(a) > y''(a)$ from Eqs. (1) and (3). By virtue of the continuity of the derivatives $z''(x)$ and $y''(x)$, the inequality $z''(x) > y''(x)$ occurs in some neighborhood, $[a,a+\epsilon]$, of the point $x = a$, where $\epsilon > 0$. Now we will prove that the inequality $z''(x) > y''(x)$ occurs for all $x\epsilon[a,b]$.

Let us assume the contrary. Let $\bar{x} > a$ be the first of the points of the interval $[a,b]$, where $z''(\bar{x}) = y''(\bar{x})$ and, consequently, $z''(x) > y''(x)$ at $a \leq x < \bar{x}$. Then, by virtue of Eq. (2), we also have $z'(x) > y'(x)$ and $z(x) > y(x)$ at $x\epsilon(a,\bar{x})$.

From Eqs. (1) and (3), we get, using the formula of finite increments,

$$z''(y) - y''(x) > F\left\{x, \lambda \int_a^x K[x,t,z(t),z'(t)]dt\right\}$$

$$- F\left\{x, \lambda \int_a^x K[x,t,y(t),y'(t)]dt\right\}$$

$$= \lambda F_u^* \int_a^x \Big\{K[x,t,z(t),z'(t)]$$

$$- K[x,t,y(t),y'(t)]\Big\}dt$$

$$= \lambda F_u^* \int_a^x \Big\{K_y^* [z(t) - y(t)]$$

$$+ K_{y'}^*[z'(t) - y'(t)]\Big\}dt \quad , \qquad (5)$$

where F_u^*, K_y^*, and $K_{y'}^*$ designate the mean values of the derivatives F_u, K_y, and $K_{y'}$, respectively.

From the inequality of Eq. (5), we will get at $x = \bar{x}$

$$z''(\bar{x}) - y''(\bar{x}) > \lambda(F_u^*)_{\bar{x}} \int_a^x \Big\{(K_y^*)_{\bar{x}} [z(t)-y(t)]$$

$$+ (K_{y'}^*)_{\bar{x}} [z'(t) - y'(t)]\Big\}dt > 0 \quad . \qquad (6)$$

The last inequality contradicts our assumption. Therefore, we have the inequality $z''(x) > y''(x)$ for all $x\epsilon[a,b]$, and, consequently, $z'(x) > y'(x)$ and $z(x) > y(x)$ at $x\epsilon[a,b]$.

Analogously proved is

Theorem 2. If the function $v(x)$ is continuous and has continuous first- and second-order derivatives in the interval $[a,b]$, it satisfies the initial conditions of Eq. (2) and the inequality

$$v''(x) < F\left\{x, \lambda \int_a^x K[x,t,v(t),v'(t)]dt\right\} \quad ,$$

$$(a \leq x \leq b) \quad . \qquad (7)$$

Then

$$v(x) < y(x) \text{ at } x\epsilon[a,b].$$

The functions $z(x)$ and $v(x)$ that satisfy the inequalities of Eqs. (3) and (7) and the initial conditions of Eq. (2) will be called, respectively, the upper and lower functions for a solution of Eq. (1).

3. It is easy to see that the inequality $z(x) > v(x)$ at $x\epsilon[a,b]$ always occurs for the upper and lower functions.

We will show first that the sets of upper and lower functions are not empty. To do this, we assume that

$$\sup |K(x,t,y,y')| = M \text{ at } (x,t,y,y')\epsilon B. \qquad (9)$$

Then it is not difficult to show that the following will, for example, be the upper function.

$$\omega(x) = y_0 + y_0' (x-a)$$

$$+\int_a^x (x-\xi)\, F[\xi,\lambda(M+1)(\xi-a)]d\xi \quad . \qquad (10)$$

It follows from Eq. (10) that $\omega(a) = y_o$ and $\omega'(a) = y_o'$. On the other hand, using the formula of finite increments, we will get

$$\omega''(x) - F\left\{x,\lambda \int_a^x K[x,t,\omega(t),\omega'(t)]dt\right\}$$

$$= F\left[x,\lambda \int_a^x (M+1)dt\right] - F\left\{x,\lambda \int_a^x K[x,t,\omega(t),\omega'(t)]dt\right\}$$

$$= \lambda F_u^* \int_a^x \left\{M + 1 - K[x,t,\omega(t),\omega'(t)]\right\}dt > 0.$$

Thus, $\omega(x)$ is shown to be the upper function. The lower function is constructed analogously.

4. We will now use the method of successive approximations to construct sequences of the upper and lower functions $\{z_n(x)\}$ and $\{v_n(x)\}$, which are uniformly convergent to the desired solution of Eq. (1).

Let us assume that the arbitrary upper and lower functions, $z_o(x)$ and $v_o(x)$, are found. As the above discussion shows, such functions can always be selected.

Then we have

$$z_o''(x) > F\left\{x,\lambda \int_a^x K[x,t,z_o(t),z_o'(t)]dt\right\}, \qquad (11)$$

$$v_o''(x) < F\left\{x,\lambda \int_a^x K[x,t,v_o(t),v_o'(t)]dt\right\}, \qquad (12)$$

and

$$z_o(x) > y(x) > v_o(x) \text{ at } x\epsilon(a,b). \qquad (13)$$

Now let us analyze the functions $z_1(x)$ and $v_1(x)$.

$$z_1(x) = y_o + y_o'(x-a)$$

$$+\int_a^x (x-\xi)F\left\{\xi,\lambda \int_a^\xi K[\xi,t,z_o(t),z_o'(t)]dt\right\}d\xi \quad , \qquad (14)$$

and

$$v_1(x) = y_o + y_o'(x-a)$$

$$+\int_a^x (x-\xi)F\left\{\xi,\lambda \int_a^\xi K[\xi,t,v_o(t),v_o'(t)]dt\right\}d\xi \ , \qquad (15)$$

$$(a \leqq x \leqq b).$$

It follows from Eq. (14) that

$$z_1(a) = y_o, \qquad z_1'(a) = y_o' \quad , \qquad (16)$$

and

$$z_1''(x) = F\left\{x,\lambda \int_a^x K[x,t,z_o(t),z_o'(t)]dt\right\} \ . \qquad (17)$$

From Eqs. (11) and (17), we have

$$z_o''(x) - z_1''(x) > 0 \text{ and } x\epsilon[a,b]. \qquad (18)$$

But, because

$$z_o(a) - z_1(a) = 0, \text{ and } z_o'(a) - z_1'(a) = 0,$$

we will get from Eq. (18)

$$z_o'(x) > z_1'(x) \text{ at } z_o(x) > z_1(x)$$

$$\text{and } x\epsilon[a,b] \ . \qquad (19)$$

Then, using Eqs. (17) and (19) and the formula of finite increments,

$$z_1''(x) - F\left\{x,\lambda \int_a^x K[x,t,z_1(t),z_1'(t)]dt\right\}$$

$$= F\left\{x,\lambda \int_a^x K[x,t,z_o(t),z_o'(t)]dt\right\}$$

$$- F\left\{x,\lambda \int_a^x K[x,t,z_1(t),z_1'(t)]dt\right\}$$

$$= \lambda F_u^* \int_a^x \left\{K_y^*[z_o(t) - z_1(t)]\right.$$

$$\left. + K_{y'}^*[z_o'(t) - z_1'(t)]\right\}dt > 0. \qquad (20)$$

It follows from Eqs. (16), (19), and (20) that $z_1(x)$ is the upper function, and that $z_0(x) > z_1(x)$. It is established analogously that $v_1(x)$ is the lower function and that $v_0(x) < v_1(x)$ at $x\epsilon[a,b]$.

Continuing this process and using the method of complete mathematical induction, we construct sequences of the upper and lower functions, $\{z_n(x)\}$ and $\{v_n(x)\}$, that satisfy the inequalities

$$z_0(x) > z_1(x) > \ldots > z_n(x) > \ldots > y(x), \quad (21)$$

and

$$v_0(x) < v_1(x) < \ldots < v_n(x) < \ldots < y(x), \quad (22)$$

$$x\epsilon(a,b),$$

where

$$z_n(x) = y_0 + y_0'(x - a)$$

$$+ \int_a^x (x-\xi)F\left\{\xi,\lambda \int_a^\xi K\,[\xi,t,z_{n-1}(t),z'_{n-1}(t)]dt\right\}d\xi\ , \quad (23)$$

and

$$v_n(x) = y_0 + y_0'(x - a)$$

$$+\int_a^x (x-\xi)F\left\{\xi,\lambda \int_a^\xi K\,[\xi,t,v_{n-1}(t),v'_{n-1}(t)]dt\right\}d\xi\ , \quad (24)$$

$$n = 1,2,\ldots, \qquad a \leq x \leq b.$$

In addition, as discussed, the inequalities $z(x) > v(x)$ and $z'(x) > v'(x)$ at $v\epsilon[a,b]$ always occur for any pair of the upper and lower functions $z(x)$ and $v(x)$.

In this manner, we will construct a monotonically decreasing sequence of the upper and lower functions. But, because we have $v_n(x) < \omega(x)$ at $x\epsilon[a,b]$ for any n, the sequence $\{v_n(x)\}$ converges. Analogously, we verify the convergence of the sequences $\{z_n(x)\}$, $\{z_n'(x)\}$, and $\{v_n'(x)\}$.

5. Let us show that the sequences $\{z_n(x)\}$, $\{v_n(x)\}$, $\{z_n'(x)\}$, and $\{v_n'(x)\}$ at $n \to \infty$ converge uniformly in the interval $[a,b]$. To do this, we introduce the following designations. Let δ_0 be the greatest of the numbers

$$\sup\,[z_0(x) - v_0(x)] \text{ and } \sup\,[z_0'(x) - v_0'(x)]$$

$$\text{at } a \leq x \leq b.$$

Let N be the greatest of the numbers

$$\sup K_y \text{ and } \sup K_{y'}$$

in the region B_0, where

$$B_0 = \{a \leq t \leq x \leq b,$$

$$v_0(x) \leq y(x) \leq z_0(x),$$

$$v_0'(x) \leq y'(x) \leq z_0'(x)\},$$

and

$$\sup \lambda F_u(x,u) = P,$$

in the region D.

Then, from Eqs. (14) and (15), we will get

$$z_1(x) - v_1(x)$$

$$= \int_a^x (x - \xi)\lambda F_u^*(\xi)d\xi \int_a^\xi \left\{K_y^*\,[z_0(t) - v_0(t)]\right.$$

$$\left. + K_{y'}^*\,[z_0'(t) - v_0'(t)]\right\}dt < 2\delta_0 PN\,\frac{(x - a)^3}{3!}\ ,$$

and

$$z_1'(x) - v_1'(x)$$

$$= \int_a^x \lambda F_u^*(\xi)d\xi \int_a^\xi \left\{K_y^*\,[z_0(t) - v_0(t)]\right.$$

$$\left. + K_{y'}^*\,[z_0'(t) - v_0'(t)]\right\} dt < 2\delta_0 PN\,\frac{(x - a)^2}{2!}\ .$$

Analogously, from Eqs. (23) and (24) at n = 2, we will get

$$z_2(x) - v_2(x)$$

$$= \int_a^x (x-\xi)\lambda F_u^*(\xi)d\xi \int_a^\xi \left\{K_y^* [z_1(t) - v_1(t)]\right.$$

$$\left. + K_{y'}^*[z_1'(t) - v_1'(t)]\right\} dt$$

$$< 2\delta_0 P^2 N^2 \frac{(x-a)^5}{5!}\left(\frac{x-a}{6} + 1\right) ,$$

and

$$z_2'(x) - v_2'(x)$$

$$= \int_a^x \lambda F_u^*(\xi)d\xi \int_a^\xi \left\{K_y^* [z_1(t) - v_1(t)]\right.$$

$$\left. + K_{y'}^*[z_1'(t) - v_1'(t)]\right\} dt$$

$$< 2\delta_0 P^2 N^2 \frac{(x-a)^4}{4!}\left(\frac{x-a}{5} + 1\right) .$$

Assuming that $\frac{b-a}{5} + 1 = L$, we will have

$$z_2(x) - v_2(x) < 2\delta_0 L P^2 N^2 \frac{(x-a)^5}{5!} , \qquad (25)$$

and

$$z_2'(x) - v_2'(x) < 2\delta_0 L P^2 N^2 \frac{(x-a)^4}{4!} . \qquad (26)$$

Using the method of complete mathematical induction, we will obtain

$$z_n(x) - v(x) < \frac{2\delta_0}{L\sqrt{LPN}} \frac{[\sqrt{LPN}\,(x-a)]^{2n+1}}{(2n+1)!} , \qquad (27)$$

and

$$z_n'(x) - v_n'(x) < \frac{2\delta_0}{L} \frac{[\sqrt{LPN}(x-a)]^{2n}}{(2n)!} . \qquad (28)$$

$$n = 1,2,3, \ldots$$

From these inequalities, it follows, in turn, that $z_n(x) - v_n(x) \to 0$ and $z_n'(x) - v_n'(x) \to 0$ at $n \to \infty$ uniformly in the interval [a,b]. Similarly, $z_n(x) \to \overline{y}(x)$, $v_n(x) \to \overline{y}(x)$, $z_n'(x) \to \omega(x)$, and $v_n'(x) \to \omega(x)$ at $n \to \infty$ uniformly in the interval [a,b].

However, because the upper and lower functions are continuous in the interval [a,b], the limiting function $\overline{y}(x)$ is also continuous in the same interval. Analogously, $\omega(x)$, $x \in [a,b]$ is a continuous function.

Turning to the limit in Eq. (23) and the equation

$$z_n'(x) = y_0' + \int_a^x F\left\{\xi, \lambda \int_a^\xi K[\xi,t,z_{n-1}(t)]dt\right\} d\xi ,$$

we will get

$$\overline{y}(x) = y_0 + y_0'(x-a)$$

$$+ \int_a^x (x-\xi) F\left\{\xi, \lambda \int_a^\xi K[\xi,t,\overline{y}(t),\omega(t)]dt\right\} d\xi ,$$

and

$$\omega(x) = y_0' + \int_a^x F\left\{\xi, \lambda \int_a^\xi K[\xi,t,\overline{y}(t),\omega(t)]dt\right\} d\xi .$$

Hence it follows that

$$\omega(x) = \overline{y}'(x) ,$$

and

$$\overline{y}''(x) + F\left\{x, \lambda \int_a^x K[x,t,\overline{y}(t),\overline{y}'(t)]dt\right\} .$$

Because the function $\overline{y}(x)$ satisfies the initial conditions $\overline{y}(a) = y_0$ and $\overline{y}'(a) = y_0'$, it is the desired solution of Eq. (1).

6. Finally, we will show that our solution of Eq. (1) which satisfies the initial conditions of Eq. (2) is unique. Indeed, we will assume that two solutions, $\overline{y}_1(x)$ and $\overline{y}_2(x)$, exist. Let us designate by δ the greatest of the numbers

$$\sup |\overline{y}_1(x) - \overline{y}_2(x)| \text{ and } \sup |\overline{y}_1'(x) - \overline{y}_2'(x)| ,$$

in the interval [a,b]. Then we will get, completely analogously to the foregoing,

$$|\overline{y}_1(x) - \overline{y}_2(x)| < \frac{2\delta}{L\sqrt{LPN}} \frac{[\sqrt{LPN}(x-a)]^{2m+1}}{(2m+1)!} ,$$

$$(m = 1,2, \ldots) \qquad (29)$$

From the inequality of Eq. (29), proceeding to the limit at $m \to \infty$, we will obtain

$$|\bar{y}_1(x) - \bar{y}_2(x)| \equiv 0 \text{ at } a \leqq x \leqq b.$$

Thus,

$$\bar{y}_1(x) \equiv \bar{y}_2(x) \text{ at } a \leqq x \leqq b.$$

Thus we have proved the existence of a unique solution of Eq. (1) which satisfies the initial conditions of Eq. (2) in any finite interval [a,b].

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