LA-4836

c. 3

Correspondence between the Time- and Criticality-Eigenvalue Problems for a Bare-Slab Reactor

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.95

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

LA-4836
UC-32
ISSUED: December 1971

Correspondence between the Time- and Criticality-Eigenvalue Problems for a Bare-Slab Reactor

by

W. L. Hendry

CORRESPONDENCE BETWEEN THE TIME- AND CRITICALITY-

EIGENVALUE PROBLEMS FOR A BARE-SLAB REACTOR

by

W. L. Hendry

ABSTRACT

The time- and criticality-eigenvalue problems are discussed, and a correspondence between time and criticality eigenvalues (and eigenfunctions) is established. The "critical flux" and lowest time eigenfunction are shown to be positive, and the associated eigenvalues are shown to be simple.

I. INTRODUCTION

Hundreds of papers have been published which present analytical studies of the neutron transport equation for idealized nuclear reactor models.[1] In the inevitable conflict between tractability and generality, the one-velocity, bare, homogeneous-slab model has frequently been chosen. This is the only model that retains continuous space, angle, and time dependence, and is at the same time almost completely understood. The omission of energy dependence is unfortunate, but is at least consistent with the current tendency among writers of transport codes to put primary emphasis on refinements in the treatment of the space and angle variables, and to carry energy dependence along by the multigroup approximation.

In saying that the transport equation for a slab is "almost completely understood," we mean that most of the important facts exist in the literature. Actually, it seems that relatively few reactor physicists avail themselves of this information. The purpose of this note is to mention briefly some of the important work that has been done on this problem, and to apply it to a discussion of the association between the criticality- and time-eigenvalue problems.

II. PREVIOUS WORK ON TRANSPORT IN A BARE SLAB

The first (and most) important analysis appeared in two papers by Lehner and Wing,[2,3] and may also be found in Wing's book.[4] Lehner and Wing found the spectrum of the transport operator, proved the existence and uniqueness of the solution to the initial-value problem, and displayed the general form that this solution takes. A characteristic of these papers is the fact that the proofs of the various theorems contain a great deal of information that the authors did not choose to state formally. The result is, of course, that these facts are not very well known.

The work of Lehner and Wing gave a nearly complete mathematical resolution of the initial-value problem, but it did not offer a convenient basis for computations. Moreover, it was difficult to see how the standard approximate solutions, such as that given by diffusion theory, differed from the exact solution. Progress in this direction had to await the development of a new tool, Case's method of singular eigenfunctions.[5,6] Using a combination of Case's method and the abstract results of Lehner and Wing, Bowden and Williams completed the analysis of the initial-value problem.[7] They showed that asymptotic diffusion theory provided a lowest order approximation to the exact solution, demonstrated how the solution could be consistently improved, and developed algorithms by which the eigenfunctions and eigenvalues could be calculated.

In another application of Case's method, Mitsis solved the critical problem for a slab.[8,9] Again, the rigorous foundation for Mitsis' results is to be found in the work of Lehner and Wing, although Mitsis made no reference to those papers. As in the time-dependent problem, asymptotic diffusion theory was shown to be a natural lowest order approximation, and ways in which this approximation could be improved were indicated.

In the years since the above works, many extensions have appeared. Inhomogeneous cross sections, anisotropic scattering, several energy groups, and various boundary conditions have all been treated with various degrees of success and rigor. Of special note is Kaper's inclusion of delayed neutrons in the time-dependent problem.[10] As a rule, significant departures from the simple problem posed by Lehner and Wing yield a drastic reduction in the number of results obtained, and those results that are obtained are no longer very "nice." Unfortunately, only the nice results are easily remembered and readily used to bolster the physicist's understanding of real systems.

III. EIGENVALUE PROBLEMS

Consider a slab extending from -a to +a on the x-axis, and surrounded by vacuum. Let the mean number of secondary neutrons emitted per collision be denoted by c, and put the velocity and total cross section equal to unity. Define the following operators:

$$L\psi = -\mu \frac{\partial \psi}{\partial x} - \psi ; \qquad (1)$$

$$S\psi = \int_{-1}^{+1} \psi(x,\mu)d\mu. \qquad (2)$$

Physically, the criticality problem is to find the smallest number, c, and the corresponding everywhere-positive flux, $\varphi(x,\mu)$, such that

$$(L + \frac{c}{2} S)\varphi = 0, \qquad (3)$$

subject to the boundary conditions

$$\varphi(\pm a,\mu) = 0, \quad \mu \lessgtr 0. \qquad (4)$$

This leads us to the "criticality-eigenvalue" problem. Find eigenvalues γ_n and corresponding nontrivial eigenfunctions $\varphi_n(x,\mu)$ such that

$$(L + \frac{\gamma_n}{2} S)\varphi_n = 0, \qquad (5)$$

subject to

$$\varphi_n(\pm a,\mu) = 0, \quad \mu \lessgtr 0. \qquad (6)$$

The initial-value problem is to find the flux $\psi(x,\mu,t)$ satisfying the equation

$$\frac{\partial \psi}{\partial t} = L\psi + \frac{c}{2} S\psi, \qquad (7)$$

subject to the boundary condition

$$\psi(\pm a,\mu,t) = 0, \quad \mu \lessgtr 0 \quad t \geq 0, \qquad (8)$$

and the initial condition

$$\psi(x,\mu,0) = f(x,\mu), \qquad (9)$$

where f is required to satisfy the same boundary condition as ψ. If we put

$$\overline{\psi}(x,\mu,\lambda) = \int_0^\infty e^{-\lambda t} \psi(x,\mu,t)dt, \qquad (10)$$

then a Laplace transformation of Eq. (7) yields

$$(L + \frac{c}{2} S)\overline{\psi} = \lambda\overline{\psi} - f. \qquad (11)$$

This leads us to the "time-eigenvalue" problem. Find eigenvalues λ_n and corresponding nontrivial eigenfunctions $\chi_n(x,\mu)$ such that

$$(L + \frac{c}{2} S)\chi_n = \lambda_n\chi_n, \qquad (12)$$

subject to

$$\chi_n(\pm a,\mu) = 0, \quad \mu \lessgtr 0. \qquad (13)$$

To be precise, we need to introduce a function space, together with specifications for the domains of our operators. However, to avoid introducing too much technical jargon at this point, we defer this to the last section, where it is needed in the proof of a theorem. Here we proceed formally, and simply inform the mathematically minded reader that the justifications for our statements are to be found (sometimes implicitly) in Ref. 4. For our present purposes, let it suffice to say that our underlying space is a Hilbert space, $\mathcal{H}$, of functions that are square-integrable in the two variables (x,μ). Note that this space is selected for mathematical convenience, rather than physical appropriateness. Physically, it would make more sense to demand that our functions be merely integrable, because it is only integrals of the flux weighted with cross sections that are ever observed.

To study the criticality-eigenvalue problem, it is convenient to put

$$\rho_n(x) = \int_{-1}^{+1} \varphi_n(x,\mu)d\mu = S\varphi_n. \qquad (14)$$

Using Eq. (5), it is easy to solve for φ_n in terms of ρ_n:

$$\varphi_n(x,\mu) = \frac{\gamma_n}{2\mu} \int_{\mp a}^{x} \exp[-(x-y)/\mu]\rho_n(y)dy, \quad \mu \gtrless 0. \qquad (15)$$

Integrating this equation over μ yields, after some manipulations,

$$\rho_n(x) = \frac{\gamma_n}{2} \int_{-a}^{a} E_1(|x-y|)\rho_n(y)dy, \qquad (16)$$

where

$$E_1(z) = \int_0^1 \frac{e^{-\frac{z}{\mu}}}{\mu} d\mu, \qquad (17)$$

is the exponential integral of the first kind. Lehner and Wing showed that the kernel E_1 is square-integrable and positive definite. It follows[11] that there exists a denumerable infinity of positive eigenvalues,

$$0 < \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots, \qquad (18)$$

and eigenfunctions, ρ_n, satisfying Eq. (16). From Eqs. (14) and (15) it follows that there is a one-to-one correspondence between the ρ_n and the φ_n. The criticality-eigenvalue problem, Eqs. (5) and (6), is therefore resolved. (Methods for computing the eigenvalues and eigenfunctions have been given by Mitsis.[8]) We note in passing that while the ρ_n are complete in $\mathcal{L}^2[-a,+a]$, the functions φ_n are not complete in $\mathcal{H}$.[12] Going back to our original physical problem, Eqs. (3) and (4), we see that it remains to show that the eigenvalue γ_0 is simple, that is, the inequality $\gamma_0 \leq \gamma_1$ is strict: $\gamma_0 < \gamma_1$. Further, we want to show that the unique eigenfunction φ_0 can be chosen nonnegative. Positivity theorems such as this have been proven for a great many reactor models, but a fairly extensive search failed to turn up the result for a bare slab. Therefore the theorem, which is an easy corollary to a known theorem in mathematics, is proven in the next section. For the remainder of this section, we shall assume the validity of this fact: the eigenvalue γ_0 is simple, $0 < \gamma_0 < \gamma_1$, and the corresponding unique eigenfunction can be chosen nonnegative.

Now the criticality problem, Eqs. (3) and (4), is resolved. One chooses $c = \gamma_0$.

The time-eigenvalue problem has been discussed in detail by Lehner and Wing.[2,3,4] Rather than simply stating their results, we reproduce part of their argument here, because it will be needed in establishing the desired association between the critical- and time-eigenvalue problems. Lehner and Wing deal with a slightly different operator from our L, but we will restate their results so that they apply here. First, they show that there are regions of the complex λ-plane in which the eigenvalues cannot lie. There can be no eigenvalues to the left of the line $\mathrm{Re}\lambda = -1$, and no eigenvalue can have an imaginary part different from zero. Moreover, $\lambda = -1$ cannot be an eigenvalue. Thus, all eigenvalues must lie on the real axis to the right of the point $\lambda = -1$.

As in the case of criticality eigenvalues, it is useful to first convert the system, Eqs. (12) and (13), to an integral equation. The result is

$$\omega_n(x) = \frac{c}{2} \int_{-a}^{a} E_1[(1 + \lambda_n)|x - y|]\omega_n(y)dy, \qquad (19)$$

where we have put

$$\omega_n(x) = \int_{-1}^{+1} \chi_n(x,\mu)d\mu. \qquad (20)$$

To find those values of λ_n for which Eq. (19) has nontrivial solutions, it is convenient to study an auxiliary problem. For fixed $\beta > 0$, find those eigenvalues, σ_n, for which

$$\sigma_n \zeta_n(x) = \int_{-a}^{a} E_1(\beta|x - y|)\zeta_n(y)dy, \tag{21}$$

has nontrivial solutions ζ_n. The kernel is again square-integrable and positive definite, and there exist a denumerable infinity of positive eigenvalues,

$$0 < \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots . \tag{22}$$

Again, as is shown in the next section, the second inequality is strict:

$$\sigma_0 < \sigma_1. \tag{23}$$

Lehner and Wing were able to discuss the behavior of the σ_n as functions of $\beta > 0$ quite thoroughly. They showed that $\sigma_0 \to +\infty$ as $\beta \to 0$ and $\sigma_n \to k_n$ with $0 < k_n < \infty$, as $\beta \to 0$ for $n > 0$. As $\beta \to \infty$, $\sigma_n \to 0$ for every n. Between these two limits, all the σ_n are strictly decreasing functions of β. This is pictured in Fig. 1. Also pictured in Fig. 1 is a graphical method of determining the λ_n. Because c is fixed, one merely draws a horizontal line a distance 2/c above the β axis. The intercepts with the graphs of the σ_n yield values of β for which nontrivial solutions exist for a given c. Call these β_n. Then,

$$\lambda_n = \beta_n - 1. \tag{24}$$

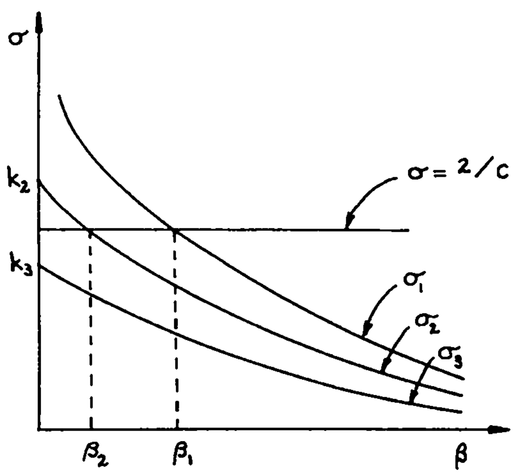

Fig. 1. Determination of time eigenvalues.

Note that for any $c > 0$ there will be only a finite number of λ_n's. Lehner and Wing also proved that the half-plane, $\mathrm{Re}\lambda \leq -1$, is continuous spectrum. Thus, as c increases, eigenvalues "emerge" from the continuous spectrum and then move to the right along the real axis.

Now we show that there is a kind of one-to-one correspondence between the time eigenvalues and the criticality eigenvalues. Referring to Fig. 1, we see that for very small $c > 0$ there will be only one time eigenvalue, λ_0, lying just to the right of $\lambda = -1$ in the complex λ plane. As c increases so will λ_0, and it will eventually take the value zero. We state that the corresponding value of c will then be just γ_0, the lowest criticality eigenvalue. If not, we would have either $0 < c < \gamma_0$ or $\gamma_0 < c$. In the former case, Eq. (12) would have a nontrivial solution with its RHS equal to zero, implying a criticality eigenvalue of less than λ_0, which is impossible. In the latter case, Eq. (5) would have a nontrivial solution with $\lambda_0 < 0$, implying a time eigenvalue greater than λ_0, again impossible. We also have

$$\rho_0(x) = \omega_0(x), \tag{25}$$

when $\lambda_0 = 0$. Hence, owing to the one-to-one correspondence between ρ_n and φ_n, and between ω_n and χ_n,

$$\varphi_0(x,\mu) = \chi_0(x,\mu), \tag{26}$$

when $\lambda_0 = 0$.

Increasing c still further, and with similar reasoning for λ_1, λ_2, ..., etc., we find that λ_n is zero when $c = \gamma_n$, and that

$$\rho_n(x) = \omega_n(x), \tag{27}$$

$$\varphi_n(x,\mu) = \chi_n(x,\mu), \tag{28}$$

when $\lambda_n = 0$. (If two or more eigenvalues are equal, the validity of these equations depends on our having made the same ordering of criticality and time eigenfunctions.) Of course, as λ_n moves to the right of $\lambda = 0$, these equalities will no longer be valid.

IV. PROOFS

We can no longer avoid a careful statement concerning our space, and the domains of our operators in the space.

Let $\mathcal{H}$ be the Hilbert space of complex-valued functions $f(z,\mu)$ defined and Lebesque square-integrable over the rectangle $\{x|-a \le x \le +a\} \times \{\mu|-1 \le \mu \le +1\}$. Define the inner product of two elements f,g in $\mathcal{H}$ by the relation

$$(f,g) = \int_{-a}^{a} dx \int_{-1}^{+1} d\mu \; f(x,\mu)\overline{g(x,\mu)} \tag{29}$$

(the bar means complex conjugation), and define the norm of f by

$$\|f\| = (f,f)^{\frac{1}{2}}. \tag{30}$$

Put

$$B = L + \frac{\gamma_0}{2} S. \tag{31}$$

The domain of the operator B, $\mathcal{D}(B)$, is a linear manifold of functions $f(z,\mu)$ in $\mathcal{H}$ satisfying the following conditions.

1. $f(z,\mu)$ is absolutely continuous in z for every $\mu \in [-1,+1]$ and $\partial f/\partial z \in \mathcal{H}$.
2. $f(z,\mu)$ is integrable in y for every $z \in [-a,+a]$ and $\int_{-1}^{+1} f d\mu \in \mathcal{H}$.
3. $f(\pm a,y) = 0$ for $\mu \lessgtr 0$.

Now we can state and prove the theorem mentioned in the last section.

THEOREM. The equation $B\varphi = 0$ has one, and only one, solution $\varphi_0 \in \mathcal{D}(B)$. This function can be chosen everywhere nonnegative, and is then positive almost everywhere.

proof. We stated in the last section that the functions $\varphi_n(x,\mu)$ were in one-to-one correspondence with the function $\rho_n(x)$, satisfying Eq. (16). Given ρ_n, we may recover φ_n using Eq. (15). To go from Eq. (5) to Eq. (16) via Eq. (15) rigorously requires some attention to details, including a justification of a change of order in an iterated integral. We omit these details here, because Wing[4] has given them in only slightly different notation.

Therefore we shall begin by showing that the smallest eigenvalue, γ_0, of Eq. (15) is simple, and the corresponding eigenfunction, ρ_0, is everywhere positive. To do this we use an extension of Jentsch's theorem on integral equations with positive kernels.[13] This extension states that if our kernel is measurable and square-integrable, and if for each $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that the iterated kernel $K^{(N)}(x,y)$ takes the value zero on a set of measure not greater than ϵ, then the following are true. The integral equation has a unique nonnegative eigenfunction, the corresponding eigenvalue is less in modulus than any other eigenvalue, the eigenfunction is positive almost everywhere, and the eigenvalue is simple.

That $E_1(|x - y|)$ is measurable and square-integrable was shown by Lehner and Wing.[2,3] The second condition is easily verified by putting $N = 1$ and noting that $E_1^{(1)} = E_1$ is everywhere positive (see Eq. (17) with $z > 0$).

Therefore the conclusions of the theorem of Krein and Rutman hold for $\rho_0(x)$, γ_0. In fact, one can show that $\rho_0(x)$ is everywhere positive. This is shown by proving that $\rho_0(x)$ is continuous. The proof follows from the fact that the singularity in E_1 is only logarithmic; although easy, it is a little long in its details and is omitted here. With $\rho_0(x)$ positive, it follows from Eq. (15) that $\varphi_0(x,\mu)$ is nonnegative. (It takes the value zero according to Eq. (13).) The function $\varphi_0(x,\mu)$ is then positive almost everywhere, because the set $\{(x,\mu)|\mu = 0\} \cup \{(x,\mu)|x = \pm a\}$ is of (plane) measure zero.

COROLLARY. When $c = \gamma_0$, the eigenvalue λ_0 in Eq. (12) is equal to zero, $\lambda_1 > \lambda_0$, and the eigenfunction $\chi_0(x,\mu)$ can be chosen nonnegative. It is then positive almost everywhere.

proof. This follows from the correspondence established in the last section, Eq. (26).

REFERENCES

1. W. L. Hendry, K. D. Lathrop, S. Vandervoort, and J. Wooten, "Bibliography on Neutral Particle Transport Theory," LA-4287-MS, Los Alamos Scientific Laboratory (1970).

2. J. Lehner and G. M. Wing, Commun. Pure Appl. Math. 8, 217 (1955).

3. J. Lehner and G. M. Wing, Duke Math. J. 23, 125 (1956).

4. G. M. Wing, An Introduction to Transport Theory, John Wiley & Sons, New York (1962).

5. K. M. Case, Ann. Phys. 9, 1 (1960).

6. K. M. Case and P. F. Zweifel, Linear Transport Theory, Addison-Wesley Publishing Company, Reading, Massachusetts (1967).

7. R. L. Bowden and C. D. Williams, J. Math. Phys. 5, 1527 (1964).

8. G. J. Mitsis, "Transport Solutions to the Monoenergetic Critical Problems," ANL-6787, Argonne National Laboratory (1963).

9. G. J. Mitsis, Nucl. Sci. Eng. 17, 55 (1963).

10. H. G. Kaper, J. Math. Anal. Appl. 19, 207 (1967).

11. F. G. Tricomi, Integral Equations, Interscience, New York (1965), p. 124.

12. B. Davison, Neutron Transport Theory, Oxford, London (1958), p. 435.

13. M. G. Krein and M. A. Rutman, Linear Operators Leaving Invariant a Cone in a Banach Space, American Mathematical Society Translation No. 26, pp. 77-79, American Mathematical Society, New York (1950).

KT/sv: 274 (20)