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SOME SOLUTIONS OF THE DIFFUSION EQUATION

by
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PHYSICS

ABSTRACT

Except for a slowly varying logarithmic factor, the diffusion of a perfectly ionized gas at constant temperature across a magnetic field has been given by $\frac{\partial n}{\partial t}$ being proportional to $\nabla \cdot \left(\frac{n \nabla n}{B^2}\right)$.

The purpose of this paper is to find similarity solutions to this equation for one-dimensional geometries. The two geometries considered in this paper are plane and cylindrical geometries. All such solutions to this equation of the form $n(x,t) = f\left[xh(t)\right] g(t)$ are found for the following two assumptions: B^2 may be considered as constant (as an approximation because of a low density); and B^2, because of static equilibrium, is non-trivially linearly dependent on n. In other words, $B^2 = B_o^2(1 - \beta)$ where B_o^2 is constant.

For the case of diffusion of a perfectly ionized gas with atomic number 1, the following formula for the rate of diffusion across a magnetic field has been given:

$$\text{Flux} = K\, n \nabla n \tag{1}$$

where n refers to the electron or ion density and

$$K = \frac{16\sqrt{\pi}\; e^2 c^2 \sqrt{\frac{m}{2\,k\,T}}\; \log_e (b_{max}/b_{min})}{3\,B^2}$$

where $b_{max} = \left[\frac{k\,T}{4\,\pi\, e^2\, n}\right]^{1/2}$

and $b_{min} = \max \left[\frac{e^2}{3\,k\,T}\, ;\; \frac{\hbar}{\sqrt{12\, m\, k\, T}}\right]$.

The continuity equation then gives that

$\frac{\partial n}{\partial t} = \nabla \cdot \left[K\, n \nabla n \right]$. Substituting

$B^2 = (1 - \beta)\, B_o^{\;2}$ and $16\,\pi\; n\, k\, T = \beta\, B_o^{\;2}$,

$$\frac{\partial \beta}{\partial t} = \nabla \cdot \left[\frac{L\,\beta}{1 - \beta} \nabla \beta \right] \tag{2}$$

where

$$L = \frac{e^2 c^2}{3} \sqrt{\frac{m}{2\pi k^3 T^3}} \quad \log_e (b_{max}/b_{min}).$$

Centering our attention on temperatures greater than 36 ev, set $b_{min} = \frac{\hbar}{\sqrt{12\, m\, k\, T}}$. Then

$$L = \frac{9.448 \times 10^5}{T^{3/2}} \left[5.923 + \frac{3}{2} \log_{10} T - \log_{10} B_o - \frac{1}{2} \log_{10} \beta \right]$$

where L is in cm^2/sec, T in electron volts, and B_o in gauss.

Taking "a practical case" where

$$T = 10^4 \text{ and } B_o = 10^4,$$

one gets

$$L = 7.486 \left[1 - 0.0631 \log_{10} \beta \right] \frac{cm^2}{sec} .$$

Since β is under a logarithm with a somewhat small coefficient, a not too unreasonable assumption would be that L is constant, in this case say about $7\frac{1}{2}$ cm^2/sec.

In the case of a slab symmetrical geometry, (2) takes the form

$$\frac{\partial \beta}{\partial t} = \frac{\partial}{\partial x} \left[\frac{L \beta}{1 - \beta} \frac{\partial}{\partial x} \beta \right] . \qquad (3)$$

In the case of a cylindrically symmetrical geometry, (2) takes the form

$$\frac{\partial \beta}{\partial t} = \frac{1}{x} \frac{\partial}{\partial x}\left[x \frac{L\beta}{1-\beta} \frac{\partial}{\partial x} \beta \right]. \qquad (4)$$

The only similarity solutions for either of these two equations of either the form

$$\beta(x,t) = f\left[xh(t) \right] g(t)$$

or

$$\beta(x,t) = 1 - f\left[xh(t) \right] g(t)$$

are

$$\beta(x,t) = f(x) \text{ and } \beta(x,t) = f(x/\sqrt{t}).$$

These solutions will be discussed at the end of this paper.

Define P as $\frac{L}{1-\beta}$. Then if β is small, we may consider P as constant and obtain from (3) and (4) the following equations:

$$\frac{\partial \beta}{\partial t} = P \frac{\partial}{\partial x}\left[\beta \frac{\partial}{\partial x} \beta \right] \qquad (5)$$

$$\frac{\partial \beta}{\partial t} = \frac{P}{x} \frac{\partial}{\partial x}\left[x\beta \frac{\partial}{\partial x} \beta \right]. \qquad (6)$$

These equations are much easier to manipulate and seem to yield many more similarity solutions than (1) and (2).

If $\beta(x,t) = f\left[\, xh(t) \,\right] g(t)$, then

$$\frac{\partial}{\partial t}\beta = fg' + xh'\, gf'$$

$$\frac{\partial}{\partial x}\beta = hf'g$$

$$\frac{\partial^2}{\partial x^2}\beta = h^2 f'' g.$$

Upon substituting into (5) and dividing by h^2g^2,

$$\frac{g'}{h^2g^2} f + xh \frac{h'}{h^3g} f' = P\left[\, ff'' + f'f' \,\right]. \tag{7}$$

Similarly, (6) yields

$$\frac{g'}{h^2g^2} f + xh \frac{h'}{h^3g} f' = P\left[\, \frac{ff'}{xh} + ff'' + f'f' \,\right]. \tag{8}$$

Therefore, for f to be a function of xh(t) only for either (7) or (8), it is necessary that one of the following conditions holds:

(a) $f(v) \equiv 0$

(b) $vf'(v) \equiv 0$

(c) The right hand side of the equation is zero.

(d) $\dfrac{g'(t)}{h^2(t)\ g^2(t)}$ and $\dfrac{h'(t)}{h^3(t)\ g^2(t)}$

are both constant.

Condition (a) leads to the uninteresting solution that $\beta(x,t) \equiv 0$.

Condition (b) leads to the solution that $\beta(x,t) \equiv$ constant.

Since P is positive, condition (c) for equation (7) implies that $f(v) = \sqrt{av + b}$ where a and b are constants. Then, inserting this expression for $f(\cdot)$ into equation (7), it is found that either a = 0 and $\sqrt{bg(t)}$ is constant, or $g(\cdot)$ and $h(\cdot)$ are constant, or b = 0 and h(t) is proportional to $1/g^2(t)$. In either of these three cases, $\beta(x,t)$ would be independent of t. Condition (c) for equation (8) implies that $f(v) = \sqrt{a \log bv}$ which again implies that $\beta(x,t)$ is independent of t.

Since conditions (a), (b), and (c) imply time independent solutions, it is sufficient to consider only condition (d) because condition (d) is satisfied by letting $g(\cdot)$ and $h(\cdot)$ be constant.

If $\dfrac{h'(t)}{h^3(t)\ g(t)} \equiv 0$, then $h(\cdot)$ is constant. Then $\dfrac{g'(t)}{g^2(t)}$ is constant so either $g(\cdot)$ is constant or g(t) is proportional to 1/t.

If $\dfrac{h'(t)}{h^3(t)\ g(t)} \not\equiv 0$, then g(t) is proportional to $\left(1/h^2(t)\right)'$.

Therefore, letting $\phi(t) \equiv 1/h^2(t)$, $\dfrac{\phi''}{\phi'}\dfrac{\phi}{\phi'}$ is proportional to $\dfrac{g'}{h^2\ g^2}$ which is constant. Therefore, for some constant c, $\dfrac{\phi''}{\phi'} = c\,\dfrac{\phi'}{\phi}$.

Integrating and taking the exponential of both sides of the equation, $\phi' = c_1\ \phi^c$ where c_1 is some constant not equal to zero.

If $c \neq 1$, then $\phi(t) = \text{constant}\ t^{\frac{1}{c-1}}$.

If $c = 1$, then $\phi(t) = \text{constant}\ e^{c_1 t}$.

Therefore, by remembering that $\phi(t) = 1/h^2$, g is proportional to ϕ' and that $g(\cdot)$ and $h(\cdot)$ may be scaled, we may let λ stand for a real number and express $g(\cdot)$ and $h(\cdot)$ as either

$$h(t) = t^{\lambda} \qquad\qquad g(t) = t^{-2\lambda-1}$$

or

$$h(t) = e^{\lambda t} \qquad\qquad g(t) = e^{-2\lambda t}.$$

Note that the cases where $\frac{h'}{h^3 g} \equiv 0$ are taken care of by setting $\lambda = 0$ in each of the above expressions.

Putting these values of $h(\cdot)$ and $g(\cdot)$ into equations (7) and (8), we get the following equations where $v \equiv xh(t)$:

$$-(2\lambda + 1)\, f(v) + \lambda v f'(v) = P\left[f(v)\, f''(v) + f'(v)\, f'(v) \right] \tag{9}$$

$$-2\lambda f(v) + \lambda v f'(v) = P\left[f(v)\, f''(v) + f'(v)\, f'(v) \right] \tag{10}$$

$$-(2\lambda + 1)\, f(v) + \lambda v f'(v) = P\left[\frac{f(v)\, f'(v)}{v} + f(v)\, f''(v) + f'(v)\, f'(v) \right] \tag{11}$$

$$-2\lambda\, f(v) + \lambda v f'(v) = P\left[\frac{f(v)\, f'(v)}{v} + f(v)\, f''(v) + f'(v)\, f'(v) \right]. \tag{12}$$

Equations (9) and (10) are for slab symmetry and equations (11) and (12) are for cylindrical symmetry. Equations (9) and (11) are for $h(t) = t^{\lambda}$ and equations (10) and (12) are for $h(t) = e^{\lambda t}$.

It seems that most of the solutions of these equations cannot be expressed by simple combinations of standard functions. However, if we integrate them and then divide by either $f(v)$ [equations (9) and (10)], or by $vf(v)$ [equations (11) and (12)], we may get them in a more easily computable form. Also, by inspection, some qualitative observations may be made.

If $\beta(x,t)$ for a slab is $f(xt^{\lambda})\, t^{-2\lambda-1}$, then

$$\frac{d}{dv} f(v) = \frac{1}{P}\left[\lambda v + \frac{c - (3\lambda + 1)\int f(v)\, dv}{f(v)}\right]. \qquad (9^*)$$

The integral in this equation is an indefinite integral and c denotes a constant of integration. If $c - (3\lambda + 1)\int f(v)\, dv$ is non-zero when $f(v) = 0$, then $f(\cdot)$ will have an infinite slope at that point. In fact, $f(\cdot)$ will go to zero as the square root of a function with a finite non-zero slope. Such an abrupt cutoff could be explained by walls. On the other hand, if this integral term is zero at $f(v) = 0$, $f(\cdot)$ would have a finite slope and this point could be thought of as the front of a freely flowing plasma.

A necessary and sufficient condition for symmetry in the sense that

$$f(v) = f(-v)$$

is that

$$c - (3\lambda + 1) \int f(v)\, dv = - (3\lambda + 1) \int_0^v f(\zeta)\, d\zeta \ .$$

We shall define "the constant is positive (or negative)" to mean that this integral expression is greater (or less) than for the symmetrical case. Since if $f(v) = \phi(v)$ is a solution to this equation, $f(v) = \phi(-v)$ is also a solution, we see that negative constants give the same types of solutions as do positive constants. When we speak of constants being large or small, we will be referring to their magnitude irrespective of sign.

If $\lambda > 0$, this solution (for small constants) is for contracting absorbing walls. For certain sized constants, one of the walls is stationary and for a sufficiently large constant, one of the walls is expanding, although still absorbing. For non-zero constants, the more rapidly contracting wall has a higher density of plasma built up near it.

If $\lambda = 0$, we get a symmetrical solution for a pair of stationary absorbing walls. The constant merely locates the center of symmetry.

If $-\frac{1}{3} < \lambda < 0$, the symmetric case is for expanding absorbing walls. With a proper choice of the constant, the plasma will be freely flowing on one side and absorbed by a wall on the other.

For a single stationary absorbing wall, $\lambda = -\frac{1}{4}$ and $f(v) = a^2 \sqrt{v} - v^2/6P$ where a is a constant. In this case, β average $\equiv \bar{\beta} = \frac{4}{9} \sqrt[3]{4}\beta_{max}$.

For a plasma of initial width of W_o and average β of $\bar{\beta}_o$ to flow to a width of W and average β of $\bar{\beta}$, it will take a time equal to

$$\frac{W_o^2}{18P\bar{\beta}_o}\left(\frac{W^4}{W_o^4} - 1\right) = \frac{W_o^2}{18P\bar{\beta}_o}\left(\frac{\bar{\beta}_o^2}{\bar{\beta}^2} - 1\right).$$

If $\lambda = -\frac{1}{3}$, the solution for the symmetric case is $f(v) = a^2 - \frac{v^2}{6P}$ where a is a constant. This is a case of a freely flowing plasma so that $\frac{d}{dt}\int_{-\infty}^{\infty}\beta(x,t)\,dx = 0$. In this case, $\bar{\beta} = \frac{2}{3}\beta_{max}$. The time for a parabolic "blob" of plasma to expand from an initial width and average β of W_o and $\bar{\beta}_o$ to W and $\bar{\beta}$ is

$$\frac{W_o^2}{36P\bar{\beta}_o}\left[\frac{R^3}{R_o^3} - 1\right] = \frac{W_o^2}{36P\bar{\beta}_o}\left[\frac{\bar{\beta}_o^3}{\bar{\beta}^3} - 1\right].$$

For the non-symmetric case, there is a wall on one side and β decreases as $1/|x|$ in the other direction.

If $-\frac{1}{2} < \lambda < -\frac{1}{3}$, then for a small constant, f(v) tends to zero as $1/|v|$ as $|v|$ tends to infinity. For a certain sized constant, we would get free flow in one direction and hyperbolic tapering off in the other. For a sufficiently large constant, we would have a wall on one side and this tapering off in the other direction.

If $\lambda = -\frac{1}{2}$, the symmetric case gives β as constant in both space and time. For small sized constants, we get a flow of plasma from one limiting level to another. In other words, f(v) approaches a positive constant as v tends to plus infinity, and approaches another positive constant as v tends to minus infinity. For a certain sized constant, we get a free flow from a limiting level into a vacuum. For large constants, we get a flow from a limiting value to a wall.

If $\lambda < -\frac{1}{2}$, we get a flow from an infinite β at infinity. For a small constant, f(v) approaches infinity as $|v|$ tends to infinity. For a certain sized constant, plasma is flowing freely into a vacuum. For a sufficiently large constant, the plasma is flowing from infinity into an absorbing wall. If $\lambda < -1$, the rate of increase of f(v) as v tends to infinity is greater than linear. For $-1 < \lambda < -\frac{1}{2}$, the rate is less than linear. For $\lambda = -1$, a special solution is $f(v) = av + a^2P$ where a is a constant. This gives $\beta(x,t) = ax + a^2Pt$. Therefore, the velocity of such a linear front is P times $-\partial\beta/\partial x$. It is easy to show that whenever $\beta(x,t) = 0$ and $\frac{\partial\beta}{\partial x}$ is finite and continuous, the velocity of such a freely flowing front will be $-P\frac{\partial\beta}{\partial x}$ if P is continuous.

If $\beta(x,t)$ for a slab is $f(xe^{\lambda t})\, e^{-2\lambda t}$, then

$$\frac{d}{dv} f(v) = \frac{1}{P}\left[\lambda v + \frac{c - 3\lambda \int f(v)\, dv}{f(v)}\right]. \qquad (10^*)$$

If $\lambda > 0$, this solution is for contracting absorbing walls. For non-zero constants, one wall is contracting faster than the other, and for sufficiently large constants, both walls will be moving in the same direction.

If $\lambda = 0$, then $\beta(x,t) = \sqrt{ax + b}$ where a and b are constants. This result implies that for the steady-state flow from a density of β_1 to β_2 across a distance W, the flux in terms of β will be $\frac{P}{2W}(\beta_1^2 - \beta_2^2)$.

If $\lambda < 0$, we get a flow from an infinite β at infinity. For a small constant, f(v) approaches infinity as $|v|$ tends to infinity. For a certain sized constant, plasma is flowing freely into a vacuum. For a sufficiently large constant, the plasma is flowing from infinity into an absorbing wall.

If $\beta(x,t)$ for a cylinder is $f(xt^{\lambda})\, t^{-2\lambda-1}$, then

$$\frac{d}{dv} f(v) = \frac{1}{P}\left[\lambda v + \frac{c - (4\lambda + 1)\int vf(v)\,dv}{vf(v)}\right]. \tag{11*}$$

For a continuous (positive) f at v = 0, the indefinite integral term, $c - (4\lambda + 1)\int vf(v)\,dv$, must be zero at the origin. We shall define "the constant is positive (or negative)," to mean that this term is larger (or smaller) than for the above case.

If the constant is negative, then f(v) would tend to infinity at the origin as $\sqrt{-\log v}$. This could be explained by plasma being fed in at the origin. (The ignoring of $1 - \beta$ in the denominator couldn't be explained.) Since the area per unit length of the cylinder, $x = \varepsilon > 0$, tends to zero as ε tends to zero, one would need an infinite flux at the origin to maintain a finite flow. If the constant is slightly positive, we would get a flow into an absorbing wall near the origin.

If $\lambda > 0$, this solution is for contracting absorbing walls. For certain sized negative constants, we may get a free flow towards the origin. These are the only λ's (where $h(t) = t^{-\lambda}$) where, by taking a proper sized constant, one can get a behavior at the origin which is different than one of the three cases given in the two previous paragraphs. Of course, these three cases are still realizable for $\lambda > 0$ as is true for all other λ's.

If $\lambda = 0$, this solution is for stationary absorbing walls.

If $-\frac{1}{4} < \lambda < 0$, this solution is for expanding absorbing walls.

If $\lambda = -\frac{1}{4}$, the solution for the case where the constant is zero is $f(v) = a^2 - \frac{v^2}{8P}$ where a is a constant. This is a case of a freely flowing plasma so that

$$\frac{d}{dt} \int_0^\infty 2\pi x \beta(x,t)\, dx = 0.$$

In this case, $\bar{\beta} = \frac{3}{4} \beta_{max}$. The time for a parabolic "blob" of plasma to expand from an initial radius and average β of R_o and $\bar{\beta}_o$ to R and $\bar{\beta}$ is

$$\frac{1}{16P} \left(\frac{R^2}{\bar{\beta}} - \frac{R_o^2}{\bar{\beta}_o} \right) = \frac{R_o^2}{16P\bar{\beta}_o} \left(\frac{R^4}{R_o^4} - 1 \right).$$

For a negative constant, there is an expanding absorbing wall on the outside which absorbs plasma at the same rate at which it is fed in at the origin. For a positive constant (plasma absorbed near the origin), f(v) tapers down towards zero as the function $1/v^2$ as v tends to infinity.

If $-\frac{1}{2} < \lambda < -\frac{1}{4}$, the solution for the cases where the constant is small or non-negative has f(v) tend to zero as the function $1/v^2$ as v tends to infinity. For a certain sized negative constant, plasma is freely flowing outward. For a sufficiently large negative constant, one gets an absorbing, expanding wall which does not absorb plasma as fast as it is being poured in at the origin.

If $\lambda = -\frac{1}{2}$, when the constant is zero, β is fixed in both space and time. For constants which are small or non-negative, β tends to a positive limiting value as v tends to infinity. For a certain negative constant, we get a free flow of plasma outward. For sufficiently large negative constants, we get expanding absorbing walls which do not absorb plasma as fast as it is being poured in at the origin.

If $\lambda < -\frac{1}{2}$, for constants that are non-negative or not too large, f(v) tends to infinity as v tends to infinity. For $-1 \leq \lambda < -\frac{1}{2}$, f(v) approaches infinity at less than a linear rate while for $\lambda < -1$, f(v) approaches infinity at greater than a linear rate. For a certain sized negative constant, we get free flow outward. For large negative constants, we get expanding absorbing walls which do not absorb plasma as fast as it is being poured in at the origin.

If $\beta(x,t)$ for a cylinder is $f(xe^{\lambda t})\ e^{-2\lambda t}$, then

$$\frac{d}{dv} f(v) = \frac{\lambda}{P} \left[v + \frac{c - 4 \int vf(v)\, dv}{vf(v)} \right]. \qquad (12^*)$$

If $\lambda > 0$, this solution is for contracting absorbing walls. As with $\lambda > 0$ for (11^*), a certain sized positive constant may give a free flow towards the origin.

If $\lambda = 0$, the case where the constant is zero gives β as fixed in space and time. For non-zero constants, $\beta(x,t) = \sqrt{a \log bx}$ where a and b are constants. This equation implies that for the steady-state flow from a density β_1 at radius r_1 to a density β_2 at radius r_2, the flow in terms of β will be

$$\frac{P\pi(\beta_1^{\,2} - \beta_2^{\,2})}{\left|\log\,(r_1/r_2)\right|}.$$

If $\lambda < 0$, a positive constant has plasma flowing from an infinite β at infinity to a wall. If the constant is zero, plasma still flows from an infinite β at infinity, but there is no wall at the origin. For small negative constants, plasma flows from infinite values at both the origin and infinity. For a certain sized negative constant, plasma is being fed in at the origin and freely flowing outward. For sufficiently large negative constants, plasma is being fed in at the origin and being absorbed by an expanding wall.

One may find that for both equations (3) and (4), there are similarity solutions of the form $\beta = f(x/\sqrt{t})$ and $\beta = f(x)$. These correspond to $\lambda = -\frac{1}{2}$ of equations (9) and (11) and $\lambda = 0$ of equations (10) and (12).

If we apply $\beta(x,t) = f(x/\sqrt{t})$ to equation (3), the slab equation, and integrate, we get

$$\frac{d}{dv} f(v) = \frac{1 - f(v)}{2L} \left[- v + \frac{c + \int f(v)\, dv}{f(v)} \right]$$

For a zero constant, β is fixed in space and time. For small constants, f(v) approaches a limit between 0 and 1 as v tends to plus infinity and another such limit as v tends to minus infinity. For a certain sized constant, we get free flow in one direction and a limiting value towards infinity in the other direction. For sufficiently large constants, we get absorbing walls, in some cases receding and in other cases approaching the plasma, while f(v) tends to a limiting value as v tends to infinity in the other direction. For sufficiently large constants (I don't know whether such a constant would need to be large enough to produce absorbing walls or not), f(v) approaches 1, instead of a value between 0 and 1, as v tends to infinity.

If we apply $\beta(x,t) = f(x/\sqrt{t})$ to equation (4), the cylindrical equation, and integrate, we get

$$\frac{d}{dv} f(v) = \frac{1 - f}{L} \left[- \frac{1}{2} v + \frac{c + \int v f(v)\, dv}{v f(v)} \right].$$

For a zero constant, β is fixed in space and time. For positive constants, there is an absorbing wall near the origin which is expanding into the plasma. For negative constants, $f(0) = 1$ and plasma is being fed in at the origin. For either non-negative or small, negative constants, $f(v)$ approaches a limiting value between 0 and 1 as v tends to infinity. For a certain sized negative constant, there is free flow away from the origin. For sufficiently large negative constants, there is an expanding, absorbing wall on the outer side of the plasma.

If we apply $\beta(x,t) = f(x)$ to equation (3), the slab equation, we get

$$e^{\beta}(1 - \beta) = e^{ax + b}$$

where a and b are constants. This equation gives that for the steady-state flow from a density of β_1 to β_2 across a distance W, the flux in terms of β will be $\frac{L}{W}\left(\beta_2 - \beta_1 + \log\frac{1 - \beta_2}{1 - \beta_1}\right)$.

If we apply $\beta(x,t) = f(x)$ to equation (4), the cylindrical equation, we get

$$e^{\beta}(1 - \beta) = \left(\frac{x}{R}\right)^{a}$$

where R is the radial position of an absorbing wall and a is an arbitrary constant (see Figure 1). This equation gives that for the steady-state flow from a density β_1 at radius r_1 to a density β_2 at radius r_2, the flow in terms of β will be

$$\frac{2\pi L}{\left|\log \frac{r_1}{r_2}\right|}\left(\beta_2 - \beta_1 + \log \frac{1 - \beta_2}{1 - \beta_1}\right).$$

If $r_1 = 0$, then $\beta_1 = 1$ and the above expression for flow is indeterminate. For the case of flow from the origin to an absorbing wall at $x = R$, define $\bar{\beta}$ as $\frac{2}{R^2} \int_0^R \beta x \, dx$. Choose a positive real number p (which is $\frac{2}{a}$) and set

$$\bar{\beta} = p\, e^p \int_0^1 z^{p-1} (1 - z)^2 e^{-pz} \, dz.$$

Then the flow of plasma is $\frac{4\pi L}{p}$. Inserting a few specific values for p, one may get the following results:

$$\bar{\beta}(1) = e - 2 = 0.71828$$

$$\bar{\beta}(2) = \frac{1}{4}(e^2 - 5) = 0.59726$$

$$\bar{\beta}(3) = \frac{2}{27}(e^3 - 13) = 0.52485$$

$$\bar{\beta}(4) = \frac{1}{128}(3e^4 - 103) = 0.47496$$

$$\bar{\beta}(6) = \frac{1}{1944}(5e^6 - 1223) = 0.40851.$$

If we consider the corresponding low β approximation, $\beta = c\sqrt{\log \frac{R}{x}}$ from $\lambda = 0$ of equation (12), we see that the definition of $\bar{\beta}$ is a convergent integral. Then the flow will be $8P\bar{\beta}^2$. If we set $P = \frac{L}{1 - \bar{\beta}}$, then it will be seen that this approximation overestimates the flow for p = 1, 2, 3, 4, and 6 by the following amounts respectively:

16.6%, 12.8%, 10.7%, 9.41%, 7.77%.

I strongly suspect that a statement of the following type is true: Consider one of the previously described similarity solutions of this paper and a bounded region of space. Then start with an arbitrary initial distribution of plasma in this region which will be assumed to satisfy $\frac{\partial \beta}{\partial t} = L \nabla \cdot \left(\frac{\beta \nabla \beta}{1 - \beta}\right)$ with L constant or $\frac{\partial \beta}{\partial t} = P \nabla \cdot (\beta \nabla \beta)$ with P constant, depending on whether we are considering similarity solutions to equations (3) or (4) or equations (5) or (6). Then apply proper boundary conditions to this region. Such boundary conditions might be of the form of fixing β on the boundary as is consistent with the similarity solution, and in the case of sources at the cylindrical origin, of stipulating the flow. The boundary could also be made to move in a way consistent with the similarity solution. Then the conclusion is that as time passes, the density distribution of the plasma converges to that given by the similarity solution.

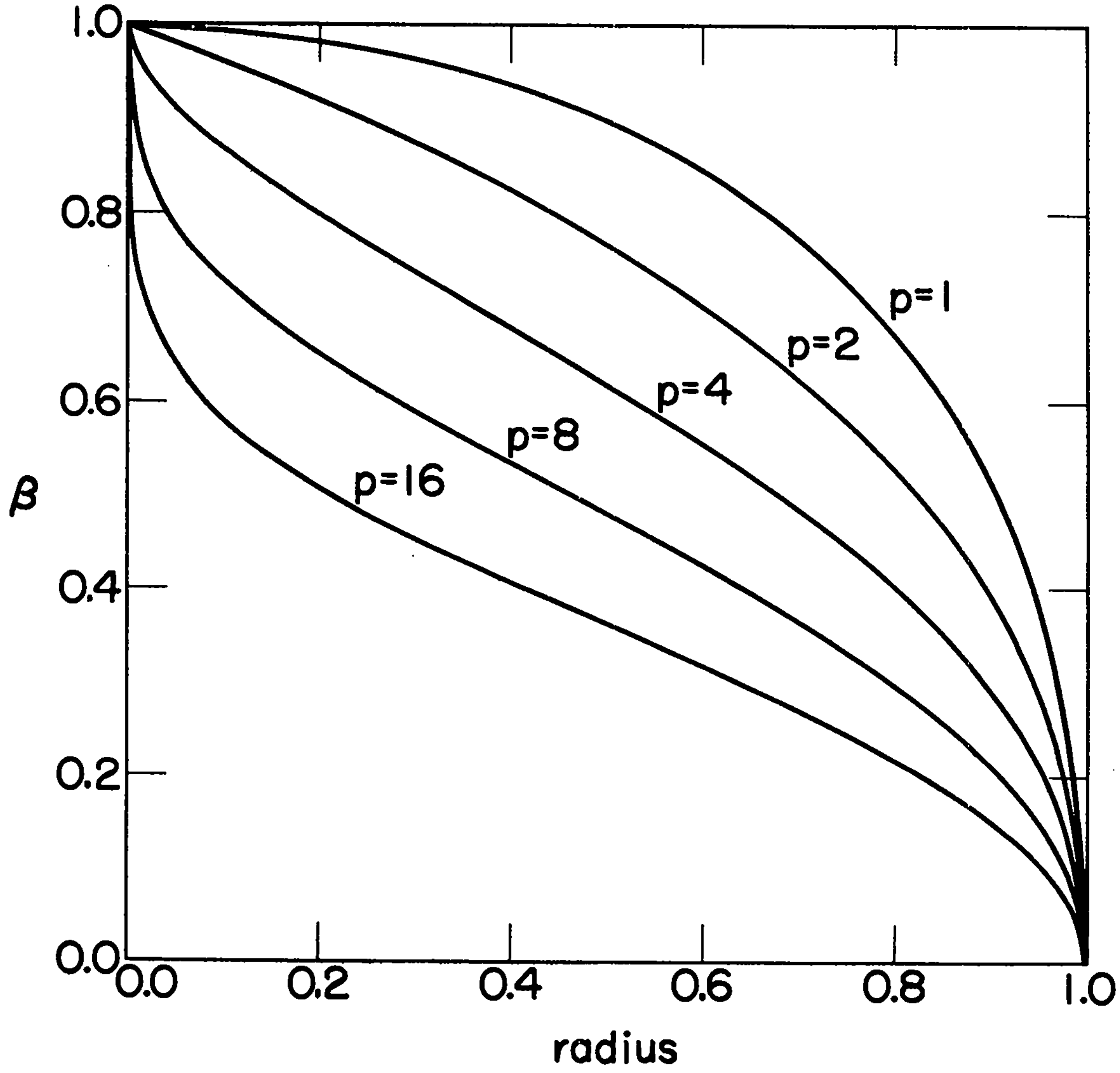

Fig. 1 Steady-State Distribution of Plasma in a Cylindrical Geometry with Source on the Axis and Absorbing Walls p = 2/a.