LA-UR 90-2744

CONF-9005237--4

OCT 0 1 1990

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

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LA-UR--90-2744

DE90 016506

SUBMITTED TO Proceedings for CNLS Conference, "Nonlinear Science, The Next Decade," May 21-25, 1990, Los Alamos, NM

Los Alamos

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

FORM NO. 836 R4

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Stability of some stationary solutions for the forced KdV equation

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The forced KdV (fKdV) equation has been established by recent studies as a simple mathematical model capable of describing the physics of a shallow layer of fluid subject to external forcing. For a particular one-parameter family of forcings which is characterized by a wave amplitude parameter for supercritical forcing distributions, exact stationary solutions are known. We study the stability of these solutions as the parameter varies. The linear stability analysis is first carried out, and we discuss the structure of the spectrum of the associated eigenvalue problem using a perturbation approach. about isolated parameter values where eigenfunctions can be expressed in closed form and are the fixed-point solutions of the fKdV equation corresponding to zero eigenvalues. The results identify a set of intervals in the parameter space corresponding to different types of manifestation of instability. In the region of the parameter space where the linear stability analysis fails to provide an answer, we have developed a nonlinear analysis to provide a sufficient condition for stability.

1 Introduction

By adopting appropriate similarity variables, the forced Korteweg-de Vries equation (fKdV) can be expressed as

$$\zeta_t - \zeta_x - \frac{3}{2}\zeta\zeta_x - \frac{1}{6}\zeta_{xxx} = P_x(x + Ft), \tag{1}$$

where $\zeta(x,t)$ refers to the free surface elevation of the liquid, while the source term P represents the bottom topography or an applied surface pressure, which is moving to the left with uniform constant velocity F. The physical units have been scaled out in (1), with ζ and x scaled by the undisturbed uniform water depth h, t by $\sqrt{h/g}$, and velocity by $\sqrt{gh}$, so that $F = 1$ corresponds to the critical speed $\sqrt{gh}$, g being the gravity acceleration[5]. This equation has drawn considerable attention in recent years, as it offers perhaps the simplest mathematical description of the interaction of a shallow layer of liquid with external forcing agents (see for instance [1−5]). Equation (1) can be derived, in analogy to the free KdV equation, under the assumption of weak nonlinearity and weak dispersion, for unidirectional wave propagation under a forcing which is sufficiently weak and moves at a near critical speed.

One of the most striking phenomena predicted by the fKdV model, which is fully confirmed by experimental observations, is the periodic emission of solitary waves propagating ahead of a steadily moving transcritical source (see e.g. [3]). However, it is known that in some cases the effects of the forcing term can remain locally confined to the forcing region, giving rise to a "pinned" solitary wave profile. Indeed, for some choices of the forcing distribution P it is possible to evaluate stationary solutions of equation (1) in closed form[6][5][7]. The problem of existence and stability of these states is one of fundamental interest (see e.g. refs. [6][5][7]), and in the case where instabilities can arise it is useful to shed some light on the basic mechanism of periodic birth of solitons in response to a forcing which is stationary in nature.

The following sections are devoted to a study of stability of one class of pinned wave profiles. For definiteness, we choose to work with an explicit form of the forcing function, the one introduced by Patoine and Warn[6] and a member of the family of solutions of Wu[5], which is supercritical and can vary with a single amplitude - parameter. However, the following considerations apply equally well to any *symmetric* (and rapidly decaying at infinity) forcing distribution. In particular, our results can be used to prove the conjectures reported in ref.[7] based on the adiabatic approximation. We first introduce in section 2 the forcing distribution with the corresponding stationary solutions, and discuss their linear stability. In section 3 we show how the Hamiltonian formalism can be exploited to establish nonlinear stability.

2 Linear stability analysis

With the forcing P and its (supercritical) velocity chosen to be

$$P(x) = a\left(k^2 - \frac{3}{4}a\right)\ \mathrm{sech}^4(kx), \quad F - 1 \equiv \frac{2}{3}k^2\ , \tag{1}$$

respectively, an elementary calculation shows that (1) has the solution(s)

$$\zeta_s(x + Ft) = a\ \mathrm{sech}^2\left[k(x + Ft)\right]\quad . \tag{2}$$

Thus, for any value of the forcing amplitude below $k^4/3$ we can have two sech^2-like solutions for ζ, one of which becomes negative in response to a negative forcing. The factor k in (2) measures the length scale of the source, and, according to the assumption of weak dispersion and near critical speed, is assumed to be a small parameter[5]. Using the scaling property of the KdV equation, one can eliminate the explicit dependence on the parameter k in (2)[5][8]: it is also convenient to rewrite (1) in a reference frame fixed with the source term. Introducing the new variables

$$x' \equiv k(x + Ft), \quad t' \equiv \frac{1}{6}k^3 t, \quad \zeta' \equiv \frac{1}{k^2}\zeta \tag{3}$$

the fKdV equation assumes the form

$$\zeta_t + 4\zeta_x - 9\zeta\zeta_x - \zeta_{xxx} = \gamma S_x, \tag{4}$$

where

$$\gamma \equiv \frac{1}{18}\alpha(12 - \alpha), \quad \alpha \equiv \frac{9a}{k^2}, \quad S(x) \equiv \text{ sech}^4(x) \ , \tag{5}$$

and we have dropped the primes for the new variables.

We now address the issue of the stability of the larger branch of the two steady state solutions given by (2) and (2) for the same forcing P with $\gamma < 2$, i.e.,

$$\zeta_s(x) = \alpha \text{ sech}^2(x), \quad \alpha \geq 6 \ . \tag{6}$$

Accordingly, we translate the dependent variable by

$$\zeta(x,t) = \zeta_s(x) + \eta(x,t), \tag{7}$$

which yields for the perturbation function η the homogenous equation

$$\eta_t + \frac{\partial}{\partial x}\left[\left(4 - \alpha \text{ sech}^2(x)\right)\eta - \frac{9}{2}\eta^2 - \eta_{xx}\right] = 0 \ . \tag{8}$$

Following the standard procedure for linear stability analysis, we linearize (8) by dropping the η^2 term and separate the variables

$$\eta(x,t) = e^{\sigma t} f(x). \tag{9}$$

Here σ and f can be complex, in which case the real part of the right hand side is understood. With this substitution, (8) reduces to the third order, non-selfadjoint, singular eigenvalue problem

$$\mathcal{L}_\alpha f \equiv \frac{d}{dx}\left[\frac{d^2 f}{dx^2} + \left(\alpha \text{ sech}^2(x) - 4\right) f\right] = \sigma f, \tag{10}$$

where the boundary conditions for f can be taken (for the discrete spectrum) as

$$f^{(n)}(x) \underset{|x|\to\infty}{\longrightarrow} 0, \quad \text{exponentially fast, for at least } n = 0,1,2\ , \tag{11}$$

$f^{(n)}$ denoting the n-th derivative of f. This type of third order eigenvalue problems have been considered recently by various authors (see e.g. refs.[6][5][9]), but very little appears to be known about the structure of their spectrum. The rest of this section will concentrate on addressing this question.

We first notice that the symmetry property of $\mathcal{L}_\alpha$ imply that if σ is an eigenvalue, then also $-\sigma$, $\pm\sigma^*$ will be eigenvalues, σ^* being the complex conjugate of σ. Hence, the stationary state is unstable as soon as an eigenvalue with a non-vanishing real part is found to exist. Furthermore, the fact that $\mathcal{L}_\alpha$ is factorized into the two operators $\frac{d}{dx}$ and

$$-(K_\alpha + 4) \equiv \frac{d^2}{dx^2} + \left[\alpha\,\mathrm{sech}^2(x) - 4\right], \tag{12}$$

can be exploited to find the eigenfunctions corresponding to the zero eigenvalue. In fact, these are the solutions (regular at infinity) of

$$(K_\alpha + 4) f(x) = 0, \tag{13}$$

and K_α is the well known Schroedinger operator with the sech^2-potential which is familiar in quantum mechanics[11]. Acceptable solutions will only occur for special values of α

$$\alpha = (2+n)(3+n), \quad n = 0,1,\dots\ , \tag{14}$$

and have alternating symmetry, starting with an even function for $\alpha = 6$. For instance, the first two eigenfunctions corresponding to zero eigenvalue are

$$\alpha_0 = 6, \quad f_0(x;\alpha_0) \;=\; B\,\mathrm{sech}^2(x), \tag{15}$$

$$\alpha_1 = 12, \quad f_0(x;\alpha_1) \;=\; B\,\mathrm{sech}^2(x)\tanh(x) \tag{16}$$

where B is an arbitrary constant. Once the special values of α at which eigenfunctions corresponding to $\sigma = 0$ of $\mathcal{L}_\alpha$ are known, we can assume that the spectrum has continuous dependence on the parameter α and look for eigenvalues close to zero using a perturbation approach. Thus, in a neighbourhood of α_n we take

$$\alpha = \alpha_n + s\epsilon, \quad n = 0, 1, \ldots, \quad 0 < \epsilon \ll 1, \quad s = \pm 1, \tag{17}$$

so that (10) becomes

$$\mathcal{L}_{\alpha_n} f(x;\alpha) = -s\epsilon \frac{d}{dx}\Big[\,\mathrm{sech}^2 x\, f(x;\alpha)\Big] + \sigma(\epsilon) f(x;\alpha), \tag{18}$$

and we expand $\sigma(\epsilon)$ and $f(x;\alpha)$ into the as asymptotic series

$$\sigma(\epsilon) = \phi_1(\epsilon)\sigma_1 + \phi_2(\epsilon)\sigma_2 + \ldots, \tag{19}$$

$$f(x;\alpha) = f_0(x) + \psi_1(\epsilon) f_1(x) + \psi_2(\epsilon) f_2(x) + \ldots \;. \tag{20}$$

The explicit form of the scaling functions ϕ_m, ψ_m is suggested by the solvability condition, i.e. orthogonality of the R.H.S. of (15) to the eigenfunction corresponding to zero eigenvalue for the adjoint of the operator $\mathcal{L}_\alpha$. It is easy to show, by integrating by parts, that these functions, denoted by $g_0(x;\alpha_n)$ can be obtained from the ones of $\mathcal{L}_\alpha$ via integration.

$$g_0(x;\alpha_n) = \int_{-\infty}^{x} f_0(y;\alpha_n)\,dy \;. \tag{21}$$

After some exploration one can see that the right scaling *is uniquely dictated by the symmetry of the eigenfunctions*, and we have $\psi_m(\epsilon) = \psi_m(\epsilon) = \epsilon^m$ for even f_0 and $\psi_m(\epsilon) = \psi_m(\epsilon) = \epsilon^{m/2}$ for odd f_0. Thus, the equation for the first order correction to $f_0(x;\alpha_0)$ (which is even) is

$$\mathcal{L}_{\alpha_0} f_1 = -s\frac{d}{dx}\Big[\,\mathrm{sech}^2 x\; f_0(x,\alpha_0)\Big] + \sigma_1 f_0(x,\alpha_0), \tag{22}$$

and the solvability condition determines σ_1,

$$s \int_{-\infty}^{+\infty} \int_{-\infty}^{x} f_0(y) \frac{d}{dx} \left[\operatorname{sech}^2(x) f_0(x, \alpha_0) \right] dy dx = \sigma_1 \int_{-\infty}^{+\infty} \int_{-\infty}^{x} f_0(y) f_0(x) dy dx \ , \tag{23}$$

or

$$\sigma(\alpha_0 + s\epsilon) = s\epsilon \, \frac{16}{5} + O(\epsilon^2) \ . \tag{24}$$

So far the perturbation approach looks quite straightforward and seems to yield the desired continuation of the fixed-point solution with the zero eigenvalue. However, integrating both sides of equation (15) and using the boundary conditions (11), one can see that

$$\sigma \int_{-\infty}^{+\infty} f(x) \, dx = 0 \ . \tag{25}$$

This is an additional constraint on eigenvalues and eigenfunctions that can be traced back to the mass conservation property of the KdV (and fKdV) equation,

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \zeta(x, t) dx = 0. \tag{26}$$

Therefore, when $\sigma = 0$ the integral of the corresponding eigenfunction may very well be different from zero, and indeed such is the case for $f_0(x)$ in our examp' However, as soon as the eigenvalue is continued away from zero, the eigenfunction integral must then vanish, inferring a singular behaviour that indicates that a regular perturbation approach is doomed to fail. The same conclusion can also be reached from (19) and the expression (20) for $\mathcal{L}_\alpha$. Imposing the boundary condition $f_1 = 0$ at $-\infty$, we have

$$f_1(x; \alpha_n) \underset{x \to +\infty}{\longrightarrow} -\frac{\sigma_1}{4} \int_{-\infty}^{+\infty} f_0 \, dx \ , \tag{27}$$

which implies that the boundary condition at $+\infty$ cannot be met by the asymptotic series (17).

The behaviour at ∞ can be corrected using the methods of matched asymptotic expansion and multiscales (cfr. [12]). The idea is to fit in a "boundary layer" when x is of order $O(\epsilon^{-1})$. We will not go into the details of this analysis, which are reported in ref.[8], and only quote the result for the uniform expansion of $f(x;\alpha)$, to order $O(\epsilon)$:

$$f(x;\alpha) = f_0(x;\alpha_0) + \epsilon\left\{f_1(x;\alpha_0) + \frac{\chi_0\sigma_1}{4}H(x)\left[1 - \exp\left(\frac{-\sigma_1}{4}\epsilon x\right)\right]\right\} + O(\epsilon^2), \tag{28}$$

where $\chi_0 = \int_{-\infty}^{\infty} f_0(x;\alpha_0)\,dx$, and $H(x)$ is the Heaviside step function. The explicit form of f_1 is not important at this order, only the limit (23) is, showing that (24) satisfies the boundary conditions (11) provided $\sigma_1 > 0$. Hence, only the case $s = 1$ can be accepted in (21), i.e., the branch of eigenvalues originating from zero at $\alpha = \alpha_0$ can only exist to the right of $\alpha = \alpha_0 = 6$.

The analysis for the "odd eigenfunction" cases (e.g., $\alpha = \alpha_1$) proceeds along similar lines, although the expansion must be carried to higher orders for consistency with (22). The resulting expression for the eigenvalue to order $O(\epsilon^{\frac{3}{2}})$ at $\alpha = \alpha_1 + s\epsilon$ is

$$\sigma(\alpha_1 + s\epsilon) = \epsilon^{\frac{1}{2}}\frac{16i}{\sqrt{105s}} + \epsilon\,\frac{16}{105s} + O(\epsilon^{\frac{3}{2}}) \quad , \tag{29}$$

which shows that the stationary state ζ_s is unstable for α on both sides of α_1, however, with quite different growth rates of the perturbation (9), of order $O(\epsilon)$ for $\alpha < \alpha_1$ and $O(\epsilon^{\frac{1}{2}})$ for $\alpha > \alpha_1$, respectively. Furthermore, the evolution of the perturbation can be expected from (25) to be in the form of a periodic oscillation with (slowly) growing amplitude for $\alpha > \alpha_1$ (with $s > 0$), due to the $O(\epsilon^{\frac{1}{2}})$ imaginary part of the eigenvalue. This oscillatory behaviour is associated with the forcing (i.e.,γ) becoming negative for $\alpha > \alpha_1$. We remark that the structure of the branches of eigenvalues starting at the other points α_n, $n > 1$ will repeat, with alternating order, the pattern of the first two. Indeed this pattern depends only on the symmetry of the appropriate fixed-point eigenfunctions, and this holds for any symmetric wave shape ζ_s replacing $\text{sech}^2 x$, provided it decays sufficiently fast at infinity. The considerations above are illustrated by fig.1, where the eigenvalue branches have been continued numerically using the high precision numerical scheme described in ref.[8]. This plot provides the additional information that the purely real branch originating at α_0 connects to the one going to the left of α_1, and the real part of the complex eigenvalue to the right of α_1 remains small, in fact decreases to be many orders of magnitude smaller than the imaginary part, the latter of which is monotonically increasing. Thus, in the range $\alpha_1 < \alpha < \alpha_2$ the pinned state ζ_s is weakly unstable, with a time periodic oscillation whose frequency increases with α.

3 The nonlinear stability property

In the previous section we have seen that the continuation method does not provide any eigenvalue for $\alpha < \alpha_0$. Thus, the question of the stability of the lower-amplitude branch solution for $\alpha < 6$ is still open. However, in the following we will show that it is exactly in this range that we can provide a sufficient condition for stability. The definition of stability is here taken to be in the usual Lyapunov sense, i.e., we can determine a distance in some appropriate functional space, where the dependent variable is defined, such that, if the perturbation superimposed to the stationary state is initially "close" to it according to this distance, it will remain close for all times[13].

We first notice that (4) can be derived from the Hamiltonian

$$\mathcal{H}(\zeta) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[\zeta_x^2 + \mu\zeta^2 - 3\zeta^3 - 6\gamma S\zeta\right] dx \ , \tag{1}$$

which is an integral of motion. The total variation of this functional at ζ_s, which is the same as the Hamiltonian for the η flow (8),

$$\Delta\mathcal{H} = \frac{1}{2}\int_{-\infty}^{+\infty} \left[\eta_x^2 - 9\zeta_s(x)\eta^2 + 4\eta^2\right] dx - \frac{3}{2}\int_{-\infty}^{+\infty} \eta^3 \, dx \ , \tag{2}$$

has a quadratic term in the difference $\eta \equiv \zeta - \zeta_s$ and a term which is cubic in η. The crucial observation is that the quadratic part can be positive definite for some values of the parameter α in ζ_s, so that the overall positivity of $\Delta\mathcal{H}$ is assured provided η is "small enough" (but still finite). In order to render this heuristic argument rigorous one has to ascertain the bounds on η so as to precisely define the meaning of "small enough". The proof is rather technical and follows the lines of ref.[14] for the free solitary waves; the details are reported in ref.[8].

Once the positivity of $\Delta\mathcal{H}$ is established, one can basically use this functional as "defining" a distance. The invariance of $\mathcal{H}$ with respect to time will then assure that this distance will not grow, or, in other words, that ζ will remain close to ζ_s provided it does so at time $t = 0$.

In order to establish the positivity of the quadratic part of $\Delta\mathcal{H}$, $\delta^2\mathcal{H}$ say, we notice that it can be rewritten as

$$\delta^2\mathcal{H} = (\eta, K_\alpha\eta) + 4\,(\eta,\eta) \ , \tag{3}$$

where (g, f) denotes the usual inner product $\int_{-\infty}^{+\infty} g^*(x)f(x)dx$, and K_α is the Schroedinger operator introduced in (12). From Sturm-Liouville theory, the first term can be estimated to be

$$(\eta, K_\alpha\eta) \geq \lambda_0(\alpha)(\eta,\eta) \tag{4}$$

where $\lambda_0(\alpha)$ denotes the infimum of the spectrum of K_α. Hence, $\delta^2\mathcal{H}$ will be positive provided

$$\lambda_0(\alpha) + 4 > 0. \tag{5}$$

It is exactly at $\alpha = \alpha_0 = 6$ that the "potential" $-\alpha\ \text{sech}^2(x)$ is deep enough to support a bound state, in the language of quantum mechanics, and the infimum of the K_α spectrum reaches a negative value, which coincides with -4. The positivity of $\Delta\mathcal{H}$ can therefore be established whenever $\alpha < \alpha_0$. We notice that for negative α, i.e., negative wave profiles, the spectrum of K_α is purely continuous, and coincides with the positive real axis. Thus, negative wave profiles are always stable. We remark at this point that the nonlinear analysis is quite general and can be carried out for any wave profile, not necessarily sech^2 or even symmetric, and depends solely on the spectrum of the operator K_α. In this sense it can immediately be applied to the pinned states described in ref.[7].

In summary, there are three basic regimes of different stability responses of the steady solitary wave solution to the pertinent one-parameter family of forcings considered here over the entire range of $-\infty < \alpha < \infty$. First, the steady solution ζ_s is globally stable for $\alpha < \alpha_0$ (with $\gamma < \gamma(\alpha_0)$) provided the perturbations are within the estimated bounds. Next, for $\alpha_0 < \alpha < \alpha_1$, the ζ_s wave is unstable since the purely real eigenvalue σ dictates an exponential growth, a trend which will repeat in the regions of $\alpha_{2n} < \alpha < \alpha_{2n+1}$ ($n = 1, 2, \ldots$). The third type of response occurs in $\alpha_1 < \alpha < \alpha_2$, which is weakly unstable and is characterized by a conspicuous periodic feature basically relevant to the phenomenon of periodic production of upstream-advancing solitons. A similar behavior of ζ_s is implied for $\alpha_{2n+1} < \alpha < \alpha_{2n+2}$ ($n = 1, 2, \ldots$).

If the parameter $\gamma = \gamma(\alpha)$ characterizing the forcing strength is used as the independent variable, there will be two branches of possible steady solutions. The answer to the question of uniqueness as to which one will result from the forcing and the stability of the resulting motion is thought to dependent critically on the initial conditions and the level of instability of ζ_s involved. For $\gamma > \gamma(\alpha_0)$ no stationary state can be found. These results should be compared to the ones reported in ref.[7] which are derived for a δ-function forcing distribution.

Acknowledgements

We are indebted to George Yates for helpful discussions. This work was jointly sponsored by DOE Contract W-7405-ENG-36, AFOSR DPP 8968, ONR Contract N00014-89-J1971 and NSF Grant 4 DMS-890 1440.

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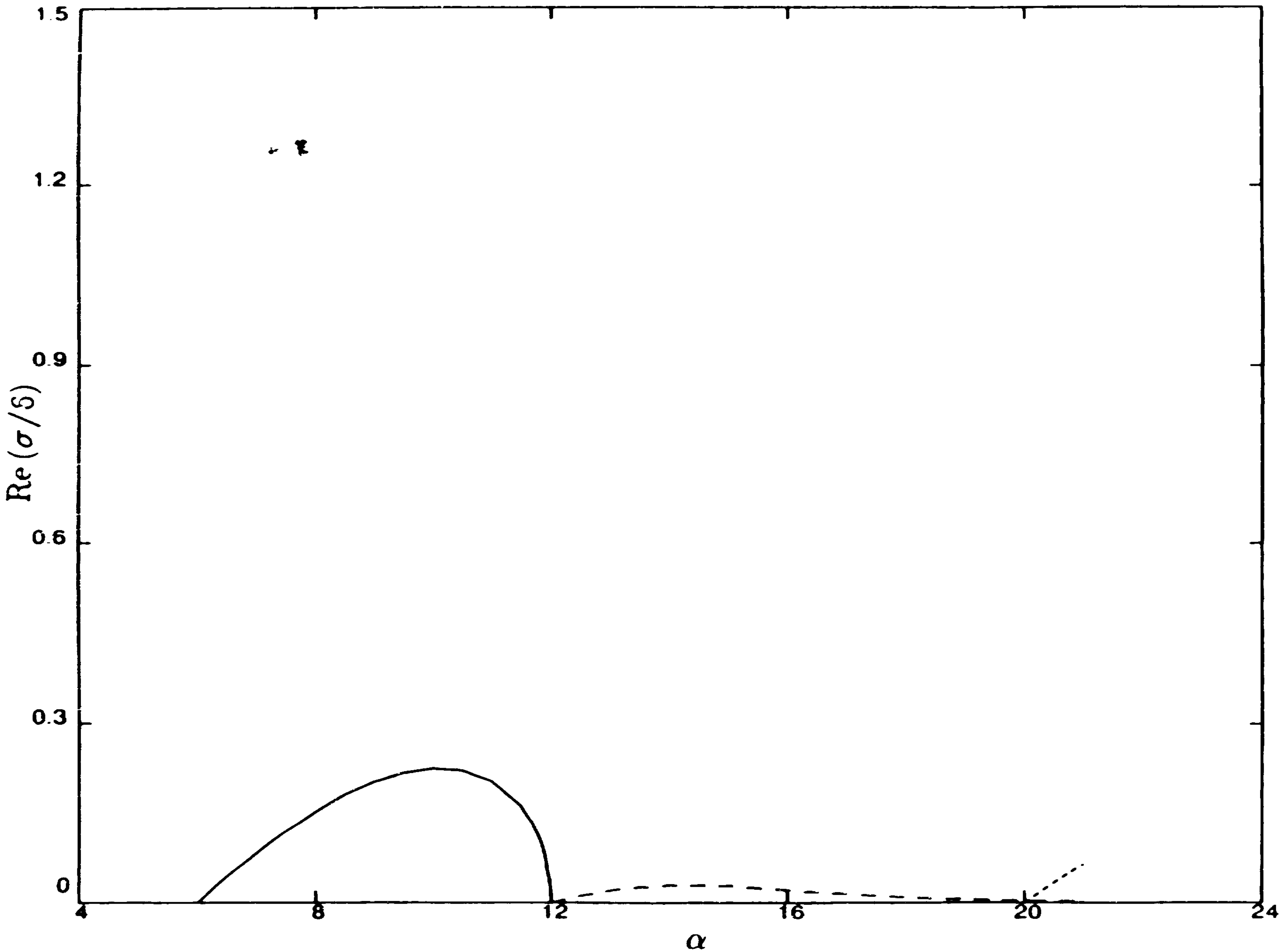
Figure captions

Figure 1. – a) The real part of $\sigma/6$ vs. α. b) Imaginary part of $\sigma/6$ vs. α.



1.5
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0.9
0.6
0.3
0
Re (σ/S)
4
8
12
16
20
24
α

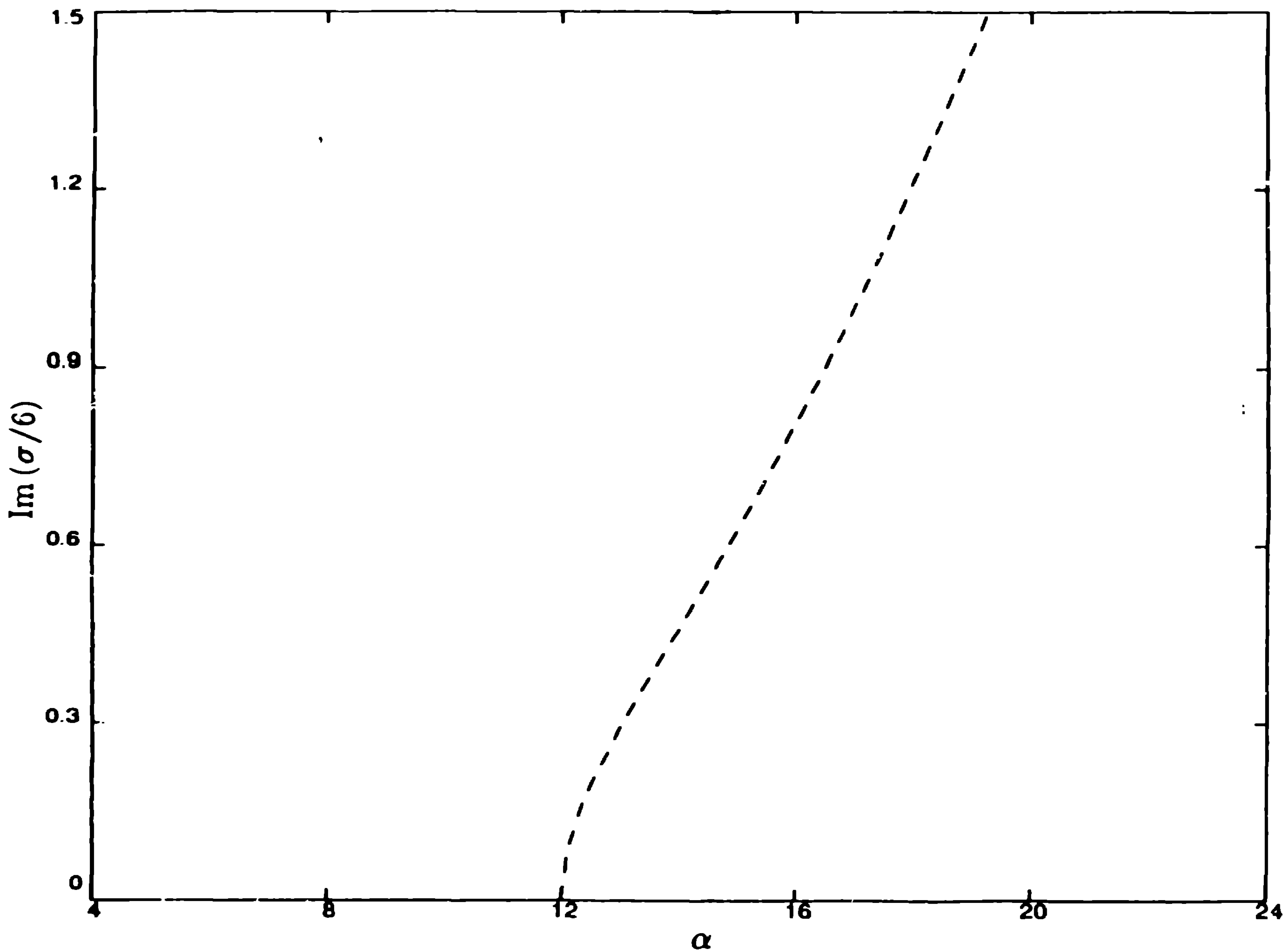

fig 1b