

*Two-Dimensional Diverging Shocks  
in a Nonuniform Medium*

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*Two-Dimensional Diverging Shocks  
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# TWO-DIMENSIONAL DIVERGING SHOCKS IN A NONUNIFORM MEDIUM

by

Roy A. Axford\*

## ABSTRACT

An analytic solution is derived from the time evolution of a two-dimensional diverging shock in  $r-z$  geometry. The shock propagates through a condensed medium characterized by a Mie-Gruneisen equation of state with a nonzero density gradient in the axial direction.

## 1. INTRODUCTION

The objective of this note is to derive an analytic solution for the time evolution of a two-dimensional diverging shock in  $r-z$  geometry. The shock is created by introducing energy  $E$  at a point and propagates through a medium characterized by a Mie-Gruneisen equation of state [1]. There is a density gradient in the axial direction which causes the shape of the shock to be nonspherical. If the density gradient is taken as zero, the analysis recovers a diverging spherical shock front. The shock strength is assumed to be infinite.

## 2. PROBLEM FORMULATION FOR A MIE-GRUNEISEN MEDIUM

Let

$$f(r, z, t) = 0 \quad (1)$$

be the equation of a diverging evolving shock front in  $r-z$  geometry at time  $t$ . The substantial time derivative of (1) can be written in the form,

$$\bar{D} \cdot \nabla f + \frac{\partial f}{\partial t} = 0 \quad , \quad (2)$$

where

$$\bar{D} = \frac{\partial r}{\partial t} \hat{e}_r + \frac{\partial z}{\partial t} \hat{e}_z, \quad (3)$$

is the shock velocity, and

$$\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{\partial f}{\partial z} \hat{e}_z \quad (4)$$

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is the gradient in cylindrical geometry. Since the gradient (4) is normal to the surface (1), an alternative form of (2) is

$$D_n |\nabla f| + \frac{\partial f}{\partial t} = 0 \quad , \quad (5)$$

where  $D_n$  is the normal component of the shock velocity.

For the case of an infinite strength shock moving into a medium at rest, the Rankine-Hugoniot relations,

$$(u_1 - D)\rho_1 = (u_0 - D)\rho_0 = m \quad , \quad (6)$$

$$p_1 + mu_1 = p_0 + mu_0 \quad , \quad (7)$$

and

$$\varepsilon_1 + \frac{p_1}{\rho_1} + \frac{1}{2}(u_1 - D)^2 = \varepsilon_0 + \frac{p_0}{\rho_0} + \frac{1}{2}(u_0 - D)^2 \quad (8)$$

are simplified with  $u_0 = 0$  and  $p_0 = 0$ . From (6) and (7) we obtain just behind the shock

$$u_1 = D \left( 1 - \frac{\rho_0}{\rho_1} \right) \quad (9)$$

and

$$p_1 = \rho_0 D u_1 \quad , \quad (10)$$

or

$$p_1 = \rho_0 D^2 \left( 1 - \frac{\rho_0}{\rho_1} \right) \quad . \quad (11)$$

With the Mie-Grüneisen equation of state,

$$p = \rho \varepsilon \Gamma(\rho / \rho_0) \quad , \quad (12)$$

where  $\Gamma(\rho / \rho_0)$  is the Grüneisen coefficient, the energy Rankine-Hugoniot relation (8) reduces to

$$\Gamma_1(\beta)\beta^2 - 2\beta[1 + \Gamma_1(\beta)] + \Gamma_1(\beta) + 2 = 0 \quad , \quad (13)$$

in which  $\beta(\rho_1 / \rho_0)$  is the compression just behind the shock. For a given Grüneisen coefficient—see, for example, Anisimov and Krachenko [1]—equation (13) is a transcendental equation for the shock compression,  $\beta$ , in terms of which we have from (11), the shock speed

$$D^2 = \left( \frac{\beta}{\beta-1} \right) \frac{p_1}{\rho_0} . \quad (14)$$

Combining (5) and (14) produces

$$\left( \frac{\partial f}{\partial r} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2 = \left( \frac{\beta-1}{\beta} \right) \frac{\rho_0(z)}{p_1} \left( \frac{\partial f}{\partial t} \right)^2 , \quad (15)$$

in which the density ahead of the shock is regarded as a function of  $z$  because of the axial density gradient.

Let  $\lambda$  denote the ratio of energy density at the shock front to the mean energy in the volume swept out by the shock front. If  $E$  is the energy introduced at a point to create the shock, and  $V(t)$  is the volume swept out by the shock front at time  $t$ , we have

$$\rho_1 \varepsilon_1 = \lambda E / V(t) . \quad (16)$$

Hence, from (12)

$$p_1 = \Gamma_1 \lambda E / V(t) , \quad (17)$$

where the energy density ratio  $\lambda$  is assumed to be a constant. By combining (15) and (17) we obtain

$$\left( \frac{\partial f}{\partial r} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2 = \left( \frac{\beta-1}{\beta} \right) \left[ \frac{\rho_0(z)V(t)}{\Gamma_1 \lambda E} \right] \left( \frac{\partial f}{\partial t} \right)^2 . \quad (18)$$

By regarding the radial coordinate at the shock front as a function of the axial coordinate and time, that is,

$$r = r(z, t) , \quad (19)$$

we can put (18) into the alternative form,

$$\frac{1}{g(z)} \left[ 1 + \left( \frac{\partial r}{\partial z} \right)^2 \right] = \left( \frac{\beta-1}{\beta} \right) \left[ \frac{\rho_0 V(t)}{\Gamma_1 \lambda E} \right] \left( \frac{\partial r}{\partial t} \right)^2 , \quad (20)$$

when the axial density variation is taken as

$$\rho_0(z) = \rho_0 g(z) . \quad (21)$$

for a specified function  $g(z)$ . To simplify (20) we introduce a new independent variable denoted by  $y$  with the chain rule,

$$\frac{\partial r}{\partial t} = \frac{\partial y}{\partial t} \frac{\partial r}{\partial y} , \quad (22)$$

and

$$\left(\frac{\beta-1}{\beta}\right)\left[\frac{\rho_0 V(t)}{\Gamma_1 \lambda E}\right]\left(\frac{\partial y}{\partial t}\right)^2 = 1 \quad . \quad (23)$$

Then

$$y = \int_0^t dt' \left[ \left(\frac{\beta}{\beta-1}\right) \frac{\Gamma_1 \lambda E}{\rho_0 V(t')} \right]^{1/2} , \quad (24)$$

and equation (20) becomes

$$\frac{1}{g(z)} \left[ 1 + \left(\frac{\partial r}{\partial z}\right)^2 \right] = \left(\frac{\partial r}{\partial y}\right)^2 . \quad (25)$$

Also, the volume swept out by the shock wave is given by

$$V(t) = \pi \int_{z_1}^{z_2} dz r^2(z, t) \quad , \quad (26)$$

where  $z_1$  is the lower intersection of the shock front with the  $z$ -axis, and  $z_2$ , the upper.

The first order partial differential equation (25) gives the shock front profile in  $r-z$  geometry for a specified axial variation  $g(z)$  of the density of the medium into which the shock is propagating. This equation can be solved by separation of variables with a sum, rather than a product, of functions in the two independent variables.

### 3. ANALYTIC SOLUTION FOR THE SHOCK FRONT

Let the shock front position which satisfies (25) be written as the sum

$$r(z, y) = F(z) + G(y) \quad . \quad (27)$$

Substituting (27) into (25) produces

$$\frac{1}{g(z)} \left[ 1 + \left(\frac{dF}{dz}\right)^2 \right] = \left(\frac{dG}{dy}\right)^2 \quad . \quad (28)$$

Since the left hand side of (28) is a function of  $z$  alone, and the right hand side, of  $y$  alone, both are equal to the separation constant taken as  $s^2$ . That is,

$$\frac{dG}{dy} = s \quad , \quad (29)$$



and

$$\left(\frac{dF}{dz}\right)^2 + 1 = s^2 g(z) \quad . \quad (30)$$

Integrating (29) and (30) gives

$$G(y) = sy \quad , \quad (31)$$

and

$$F(z) = \int_0^z dz' \left[ s^2 g(z') - 1 \right]^{1/2} \quad , \quad (32)$$

for a specified relative axial density variation. From (27), (31) and (32) there follows

$$r(z, y) = sy + \int_0^z dz' \left[ s^2 g(z') - 1 \right]^{1/2} \quad , \quad (33)$$

and

$$\frac{\partial r}{\partial s} = y + \int_0^z dz' s g(z') \left[ s^2 g(z') - 1 \right]^{-1/2} = C(s) \quad . \quad (34)$$

The number  $C(s)$  in (34) can be taken as zero on the basis of the following argument. For very small times the shock front is spherical in shape as would be the case if the shock were propagating into a uniform density medium. Hence, setting  $g(z) = 1$  in (33) and (34) yields

$$r = s y + z \sqrt{s^2 - 1} \quad , \quad (35)$$

and

$$\frac{\partial r}{\partial s} = y + \frac{zs}{\sqrt{s^2 - 1}} = C(s) \quad . \quad (36)$$

From (35) and (36) we have upon eliminating  $y$ ,

$$r = s C(s) - \frac{z}{\sqrt{s^2 - 1}} \quad . \quad (37)$$

Also from (36)

$$y^2 = C^2(s) - \frac{2zsC(s)}{\sqrt{s^2 - 1}} + \frac{z^2 s^2}{s^2 - 1} \quad . \quad (38)$$

Now set  $C(s) = 0$ ; then it follows from (38) that

$$s^2 - 1 = \frac{z^2}{y^2 - z^2} \quad . \quad (39)$$

Eliminating  $s$  from (37) with  $C(s) = 0$  and with (39) yields

$$r^2 + z^2 = y^2 \quad , \quad (40)$$

which is the equation of a sphere in cylindrical coordinates. Accordingly, equations (33) and (34) represent a spherical shock if the initial density is spatially uniform or at very small times if the initial density is not spatially uniform when  $C(s) = 0$  in (34).

An alternative form of (34) with  $C(s) = 0$  is

$$sy + \int_0^z dz' [s^2 g(z') - 1]^{1/2} + \int_0^z dz' [s^2 g(z') - 1]^{-1/2} = 0 \quad . \quad (41)$$

Hence, an alternative form of (33) is

$$r(z, y) = -\int_0^z dz' [s^2 g(z') - 1]^{-1/2} \quad . \quad (42)$$

Equations (34) with  $C(s) = 0$  and (42) give the time evolution of a nonspherical shock front for a specified axial density variation. As an example, take an exponential variation, namely,

$$g(z) = \exp(-z / z_0) \quad . \quad (43)$$

Then, equation (42) becomes

$$r(z, y) = -\int_0^z dz' [s^2 \exp(-z' / z_0) - 1]^{-1/2} \quad . \quad (44)$$

With the change in integration variable,

$$X^2 = s^2 \exp(-z' / z_0) \quad , \quad (45)$$

equation (44) is equivalent to

$$r(z, y) = 2z_0 - \int_s^{s \exp(-z/2z_0)} dX X^{-1} [X^2 - 1]^{-1/2} \quad , \quad (46)$$

which simplifies to

$$r(z, y) = 2z_0 \left[ \text{arc cos} \left( \frac{1}{s \exp(-z / 2z_0)} \right) - \text{arc cos} \left( \frac{1}{s} \right) \right] \quad . \quad (47)$$

Also, with the integration variable defined in (35) equation (34) with  $C(s) = 0$  becomes

$$\frac{\partial r}{\partial s} = y + \frac{2z_0}{s} \int_{s \exp(-z/2z_0)}^s dX X [X^2 - 1]^{-1/2} = 0 \quad , \quad (48)$$

which simplifies to

$$sy + 2z_0 \left[ (s^2 - 1)^{1/2} - (s^2 \exp(-z/z_0) - 1)^{1/2} \right] = 0. \quad (49)$$

The separation constant  $s$  can be eliminated from (47) and (49) as follows. In (47) introduce the definitions,

$$\theta_1 = \arccos \left( \frac{1}{s \exp(-z/2z_0)} \right), \quad (50)$$

and

$$\theta_2 = \arccos \left( \frac{1}{s} \right). \quad (51)$$

Then we have from (47)

$$\cos(r/2z_0) = \cos(\theta_1 - \theta_2), \quad (52)$$

which with the trigonometric addition theorem reduces to

$$\cos(r/2z_0) = \left( \frac{1}{s \exp(-z/2z_0)} \right) \left[ 1 + (s^2 - 1)^{1/2} (s^2 \exp(-z/z_0) - 1)^{1/2} \right]. \quad (53)$$

From (49) it follows that

$$1 + (s^2 - 1)^{1/2} (s^2 \exp(-z/z_0) - 1)^{1/2} = \frac{s^2(1 - w^2)}{2} + \frac{s^2 \exp(-z/z_0)}{2}, \quad (54)$$

where

$$w = y/2z_0. \quad (55)$$

Combining (53) and (54) produces

$$r(z, y) = 2z_0 \arccos \left\{ \frac{\exp(z/2z_0)}{2} \left[ 1 - \left( \frac{y}{2z_0} \right)^2 + \exp(-z/z_0) \right] \right\}. \quad (56)$$

for the shape of the nonspherical diverging shock front.

By setting  $r(z, y) = 0$  in (56), the lower point  $z_1$  and the upper point  $z_2$  of the shock front can be found. That is, these two points are given by the two solutions of the quadratic equation,

$$e^{-z/z_0} - 2e^{-2z/z_0} + 1 - w^2 = 0, \quad (57)$$

which are

$$z_1 = -2z_0 \ln \left( 1 + \frac{y}{2z_0} \right) < 0 \quad , \quad (58)$$

and

$$z_2 = -2z_0 \ln \left( 1 - \frac{y}{2z_0} \right) > 0 \quad . \quad (59)$$

The explicit time dependence in the shock front solution (56) is determined with (24) and (26). With (58) and (59) and the change in variable  $z = z_0 u$ , the volume swept out by the shock in (26) is given by

$$V = 4\pi z_0^3 \int_{-2\ln(1+w)}^{-2\ln(1-w)} du \arccos^2 \left[ \frac{\exp(u/2)}{2} (1 - w^2 + \exp(-u)) \right] \quad , \quad (60)$$

which is

$$V = 4\pi z_0^3 W(w) \quad , \quad (61)$$

where  $W(w)$  is the integral in (60). From (24) we have

$$dt = \left[ \frac{(\beta-1)\rho_0}{\beta\Gamma_1\lambda E} \right]^{1/2} \sqrt{V} dy \quad . \quad (62)$$

Combining (55) and (61) with (62) produces

$$t = \left[ \frac{16\pi z_0^5 \rho_0 (\beta-1)}{\beta\Gamma_1\lambda E} \right]^{1/2} \int_0^w dw' \sqrt{W(w')} \quad . \quad (63)$$

Equations (56) and (63) together with (55) give the time evolution of the shock front in  $r-z$  geometry. The integrals in (60) and (63) can be evaluated numerically but not in closed form. The results in (56) and (63) are limited to the time interval over which the infinite shock strength approximation holds.

## REFERENCES

1. S. I. Anisimov and V. A. Kravchenko, "Shock Wave in Condensed Matter Generated by Impulsive Load," *Z. Naturforsch.*, **40a**, 8-13 (1985).

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