LA-13497

Solutions of the Noh Problem for Various Equations of State Using Lie Groups

Los Alamos
NATIONAL LABORATORY

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36.

Edited by Patricia W. Mendius, Group CIC-1
Prepared by Lucille Maestas, Group X-HM

LA-13497

UC-705
Issued: August 1998

Solutions of the Noh Problem for Various Equations of State Using Lie Groups

*Roy A. Axford**

**Consultant at Los Alamos. Department of Nuclear Engineering, University of Illinois, 103 S. Goodwin, Urbana, IL 61801.*

Los Alamos
NATIONAL LABORATORY
Los Alamos, New Mexico 87545

CONTENTS

SOLUTIONS OF THE NOH PROBLEM FOR VARIOUS EQUATIONS OF STATE USING LIE GROUPS

by

Roy A. Axford*

ABSTRACT

A method for developing invariant equations of state for which solutions of the Noh problem will exist is developed. The ideal gas equation of state is shown to be a special case of the general method. Explicit solutions of the Noh problem in planar, cylindrical and spherical geometry are determined for a Mie-Gruneisen and the stiff gas equation of state.

1. INTRODUCTION

The general objective of this study is to determine equations of state other than that of the ideal gas for which solutions of the Noh problem [1] will exist. In the Noh problem the hydrodynamics equations are solved by reduction to ordinary differential equations written out in terms of similarity variables that can be constructed as invariant functions of a multiparameter group of point transformations. This group must be admitted by the hydrodynamics equations for the particular equation of state used. Equations of state such that the corresponding hydrodynamics equations admit a multiparameter group are called invariant equations of state. A method for developing invariant equations of state is developed in Sections 2 and 3 together with that of constructing similarity variables for the Noh problem. Explicit solutions of the Noh problem in planar, cylindrical, and spherical geometries are found in Section 4 for (1) the ideal gas, (2) a Mie-Gruneisen, and (3) the stiff gas equation of state.

2. HYDRODYNAMICS EQUATIONS IN TERMS OF THE ADIABATIC BULK MODULUS

If entropy is regarded as a function of pressure and density, the energy equation for isentropic flows is

$$\frac{DS}{Dt} = \left.\frac{\partial S}{\partial p}\right|_{\rho} \frac{Dp}{Dt} + \left.\frac{\partial S}{\partial \rho}\right|_{p} \frac{D\rho}{Dt} = 0 \quad , \tag{1}$$

where $\frac{D}{Dt}$ denotes the substantial time derivative. With the continuity equation,

$$\frac{D\rho}{Dt} + \rho \, div \, \vec{u} = 0 \quad , \tag{2}$$

*Consultant; University of Illinois, Department of Nuclear Engineering, 103 S. Goodwin, Urbana, IL 61801

the energy balance (1) can be written in the form,

$$\frac{Dp}{Dt}+A(p,\rho)\,div\,\vec{u}=0 \quad , \tag{3}$$

in which $A(p,\rho)$ is the adiabatic bulk modulus defined by

$$A(p,\rho)=-\rho\frac{\left.\frac{\partial S}{\partial \rho}\right|_p}{\left.\frac{\partial S}{\partial p}\right|_\rho} \quad . \tag{4}$$

Since the sonic speed $c^2(p,\rho)$ is given by

$$c^2(p,\rho)=\left.\frac{\partial p}{\partial \rho}\right|_s=-\frac{\left.\frac{\partial S}{\partial \rho}\right|_p}{\left.\frac{\partial S}{\partial p}\right|_\rho} \quad , \tag{5}$$

the adiabatic bulk modulus can be expressed in terms of the sonic speed by

$$A(p,\rho)=\rho\;c^2(p,\rho) \quad . \tag{6}$$

If the internal energy ε is a function of the pressure and density, the adiabatic bulk modulus and internal energy satisfy the first order partial differential equation,

$$A(p,\rho)\left.\frac{\partial \varepsilon}{\partial p}\right|_\rho+\rho\left.\frac{\partial \varepsilon}{\partial \rho}\right|_p=\frac{p}{\rho} \quad , \tag{7}$$

which can be derived as follows. By combining the first and second laws of thermodynamics, we have

$$d\varepsilon=TdS+\frac{p}{\rho^2}d\rho \quad . \tag{8}$$

Expanding the differentials yields

$$\left.\frac{\partial \varepsilon}{\partial p}\right|_\rho dp+\left.\frac{\partial \varepsilon}{\partial \rho}\right|_p d\rho=T\left[\left.\frac{\partial S}{\partial p}\right|_\rho dp+\left.\frac{\partial S}{\partial \rho}\right|_p d\rho\right]+\frac{p}{\rho^2}d\rho \quad . \tag{9}$$

Since this must be an identity in the differentials of pressure and density, equating the coefficients of these differentials gives

$$T\left.\frac{\partial S}{\partial p}\right|_\rho=\left.\frac{\partial \varepsilon}{\partial p}\right|_\rho \tag{10}$$

and

$$T\frac{\partial S}{\partial \rho}\bigg|_{p} = \frac{\partial \varepsilon}{\partial \rho}\bigg|_{p} - \frac{p}{\rho^2} \quad . \tag{11}$$

Substituting (10) and (11) into (4) produces (7). Given the equation of state,

$$\varepsilon = \varepsilon(p, \rho) \quad , \tag{12}$$

the corresponding adiabatic bulk modulus can be evaluated with (7). Conversely, given the adiabatic bulk modulus, the corresponding equation of state can be determined by solving the characteristic equations of (7), namely,

$$\frac{dp}{A(p,\rho)} = \frac{d\rho}{\rho} = \frac{\rho d\varepsilon}{p} \quad . \tag{13}$$

If the equation of state is taken in the form in which the pressure is regarded as a function of the internal energy ε and density, that is

$$p = p(\varepsilon, \rho) \quad , \tag{14}$$

then the pressure and adiabatic bulk modulus satisfy the first order partial differential equation,

$$A(p,\rho) = \rho\,\frac{\partial p}{\partial \rho}\bigg|_{\varepsilon} + \frac{p}{\rho}\frac{\partial p}{\partial \varepsilon}\bigg|_{\rho} \quad , \tag{15}$$

which can be derived as follows. An alternative form to (1) for the energy balance is

$$\rho\frac{D\varepsilon}{Dt} + p\, div\, \vec{u} = 0 \quad . \tag{16}$$

The substantial time derivative of (14) is

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \varepsilon}\bigg|_{\rho}\frac{D\varepsilon}{Dt} + \frac{\partial p}{\partial \rho}\bigg|_{\varepsilon}\frac{D\rho}{Dt} \quad . \tag{17}$$

From (16) it follows that

$$\frac{\partial p}{\partial \varepsilon}\bigg|_{\rho}\frac{D\varepsilon}{Dt} = (-1)\frac{\partial p}{\partial \varepsilon}\bigg|_{\rho}\frac{p}{\rho}\, div\, \vec{u} \quad . \tag{18}$$

By substituting (18) and the continuity equation (2) into (17), we obtain

$$\frac{Dp}{Dt} + \left[\frac{p}{\rho}\frac{\partial p}{\partial \varepsilon}\bigg|_{\rho} + \rho\,\frac{\partial p}{\partial \rho}\bigg|_{\varepsilon}\right] div\, \vec{u} = 0 \quad . \tag{19}$$

Comparing (19) with (3) yields (15). Given the equation of state (14), the corresponding adiabatic bulk modulus can be computed from (15). Given the adiabatic bulk modulus, the equation of state (14) can also be found by solving the characteristic equations (13).

Examples of the adiabatic bulk modulus include the following:

(1) For the ideal gas equation of state,

$$p = (\gamma - 1)\rho\varepsilon \quad , \tag{20}$$

the adiabatic bulk modulus from (15) is

$$A(p,\rho) = \gamma\, p \quad . \tag{21}$$

(2) For the Mie-Gruneisen equation of state,

$$p = \rho\,\varepsilon\,\Gamma(\rho) \quad , \tag{22}$$

in which $\Gamma(\rho)$ is the Gruneisen coefficient, we have

$$A(p,\rho) = p\left[\Gamma + \frac{1}{\Gamma}\frac{\partial}{\partial\rho}(\rho\Gamma)\right] \quad . \tag{23}$$

(3) For the stiff gas equation of state,

$$p = a^2(\rho - \rho_\infty) + (\gamma - 1)\rho\varepsilon \quad , \tag{24}$$

we have

$$A(p,\rho) = \gamma\, p + a^2\rho_\infty \quad . \tag{25}$$

A useful form of the Gruneisen coefficient $\Gamma(\rho)$ is that given by Anisimov and Kravchenko [2], namely,

$$\Gamma(\rho/\rho_\infty) = \frac{2}{3} + \left(\Gamma_0 - \frac{2}{3}\right)\frac{f_m^2 + 1}{f_m^2 + f^2} f \quad , \tag{26}$$

where

$$f = \rho/\rho_\infty \quad . \tag{27}$$

In terms of the adiabatic bulk modulus, the hydrodynamics equations are

(1) the continuity equation,

$$\frac{D\rho}{Dt} + \rho\, div\, \vec{u} = 0 \quad , \tag{28}$$

(2) the equation of motion,

$$\rho \frac{D\vec{u}}{Dt} + \text{grad } p = 0 \quad , \tag{29}$$

(3) and the energy balance,

$$\frac{Dp}{Dt} + A(p,\rho)\, div\, \vec{u} = 0 \quad . \tag{30}$$

3. INVARIANCE PROPERTIES OF THE HYDRODYNAMICS EQUATIONS WITH AN ARBITRARY ADIABATIC BULK MODULUS

In one-dimensional planar, cylindrical and spherical geometries, the hydrodynamics equations (28)-(30) become

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{Nu}{r}\right) = 0 \quad , \tag{31}$$

$$\rho(u_t + u\, u_r) + p_r = 0 \quad , \tag{32}$$

and

$$p_t + up_r + A(p,\rho)\left(u_r + \frac{Nu}{r}\right) = 0 \quad , \tag{33}$$

where the subscripts t and r denote partial derivatives with respect to time and the space coordinate, respectively. In (31)-(33), for planar geometry $N = 0$, for cylindrical geometry $N = 1$, and for spherical geometry $N = 2$.

The hydrodynamics equations (31)-(33) for both cylindrical and spherical geometry are invariant under the four-parameter group with the Lie algebra of generators,

$$\hat{U}_1 = \frac{\partial}{\partial t} \quad , \tag{34}$$

$$\hat{U}_2 = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} \quad , \tag{35}$$

$$\hat{U}_3 = r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u} - 2\rho\frac{\partial}{\partial \rho} \quad , \tag{36}$$

and

$$\hat{U}_4 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p} \quad , \tag{37}$$

provided that the adiabatic bulk modulus in (33) is that of the ideal gas given in (21). In the case of planar geometry, the Lie algebra is six-dimensional with the four generators (34) - (37) together with the spatial translation,

$$\hat{U}_5 = \frac{\partial}{\partial r} \quad , \tag{38}$$

and the Galilean boost,

$$\hat{U}_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \quad . \tag{39}$$

All transformations of the four-parameter group with the Lie algebra basis (34) - (37) for the ideal gas are generated by

$$\hat{U} = (e_2 + e_3) r \frac{\partial}{\partial r} + (e_1 + e_2\, t) \frac{\partial}{\partial t} + e_3\, u \frac{\partial}{\partial u}$$

$$+ e_4\, p \frac{\partial}{\partial p} + (e_4 - 2e_3) \rho \frac{\partial}{\partial \rho} \quad ; \tag{40}$$

where $e_i, i = 1,2,3,4$, are the four group parameters. To verify that the hydrodynamics equations (31) - (33) with the ideal gas adiabatic bulk modulus (21) admit the four-parameter group with generator (40), the prolongation of this generator to first order partial derivatives is evaluated and found to be given by

$$\hat{U}^{(1)} = (e_2 + e_3) r \frac{\partial}{\partial r} + (e_1 + e_2\, t) \frac{\partial}{\partial t} + e_3 u \frac{\partial}{\partial u} + e_4 p \frac{\partial}{\partial p}$$

$$+ (e_4 - 2e_3) \rho \frac{\partial}{\partial \rho} - e_2\, u_r \frac{\partial}{\partial u_r} + (e_3 - e_2) u_t \frac{\partial}{\partial u_t}$$

$$+ (e_4 - e_2 - e_3) p_r \frac{\partial}{\partial p_r} + (e_4 - e_2) p_t \frac{\partial}{\partial p_t}$$

$$+ (e_4 - e_2 - 3e_3) \rho_r \frac{\partial}{\partial \rho_r} + (e_4 - e_2 - 2e_3) \rho_t \frac{\partial}{\partial \rho_t} \quad . \tag{41}$$

By operating on the three hydrodynamics equation (31)-(33) for the ideal gas case with the prolonged group generator (41), it is found that the hydrodynamics equations are three invariant equations of the four parameter group of point transformations generated by (40).

What condition must be satisfied by adiabatic bulk modulus $A(p,\rho)$ in the energy balance (33) in order that the hydrodynamics equations (31)-(33) admit the four parameter group generated by (40)? As indicated above, (31)-(33) are invariant under (40) if the adiabatic bulk modulus is that of the ideal gas given in (21). However, are there other forms of the adiabatic bulk modulus such that the hydrodynamics equations (31)-(33) are also invariant under the four parameter group generated by (40)? For the case of an arbitrary adiabatic bulk modules, the energy equation (33) admits the group generated by (40) if the invariance condition,

$$\hat{U}^{(1)} \left[p_t + u p_r + A(p,\rho) \left(u_r + \frac{Nu}{r} \right) \right] = 0 \quad , \tag{42}$$

is satisfied when evaluated on (33). In (42) note the appearance of the generator $\hat{U}^{(1)}$ of the first prolongation. Equation (42) is the Lie derivative of the energy balance (33) which reduces to a linear first order partial differential equation for the adiabatic bulk modulus, namely,

$$e_4 p \frac{\partial A}{\partial p} + (e_4 - 2e_3)\rho \frac{\partial A}{\partial \rho} - e_4 A = 0 \quad . \tag{43}$$

If the adiabatic bulk modulus is a solution of this partial differential equation, then the corresponding hydrodynamics equations (31)-(33) will be invariant under the four parameter group generated by (40).

Also, if the adiabatic bulk modulus is a solution of (43), then similarity variables can be constructed by solving the characteristic equations of the first order partial differential equation,

$$\hat{U}F = 0 \quad , \tag{44}$$

formed with the group generator (40). These characteristic equations are

$$\frac{dr}{(e_2 + e_3)r} = \frac{dt}{e_1 + e_2 t} = \frac{du}{e_3 u} = \frac{dp}{e_4 p} = \frac{d\rho}{(e_4 - 2e_3)\rho} \quad . \tag{45}$$

Solutions of these characteristic equations provide similarity variables in terms of which the hydrodynamics equations (31)-(33) can be reduced to ordinary differential equations when the adiabatic bulk modulus is a solution of (43). The corresponding equation of state for which the hydrodynamics equations (31)-(33) will have similarity solutions can be found by integrating (7) or (15).

The method described above provides similarity variables to reduce the hydrodynamics equations (31)-(33) to ordinary differential equations with equations of state whose adiabatic bulk modulus is a solution of (43) for three classes of problems, namely, (1) the Noh problem, (2) the Guderley converging shock problem, and (3) the blast wave diverging shock problem. In the following sections the Noh problem will be solved for various equations of state.

4. SOLUTIONS OF THE NOH PROBLEM FOR VARIOUS EQUATIONS OF STATE

The hydrodynamics equations (31)-(33) are invariant under the four parameter group with generator (40) when the adiabatic bulk modulus $A(p,\rho)$ in the energy balance (33) is that (1) for the ideal gas equation of state given in (21), (2) for the Mie-Gruneisen equation of state given in (23), and (3) for the stiff gas equation of state given in (25). These results are proved as follows. Set $e_4 = 2e_3$ in the adiabatic bulk modulus invariance condition (43) to get

$$p \frac{\partial A}{\partial p} + 0 \frac{\partial A}{\partial \rho} = A \quad . \tag{46}$$

The characteristic equations of (46), namely,

$$\frac{dp}{p}=\frac{d\rho}{0}=\frac{dA}{A} \quad , \tag{47}$$

have the solutions,

$$\rho = \text{constant} \quad , \tag{48}$$

and

$$\frac{A(p,\rho)}{p} = \text{constant} \quad . \tag{49}$$

Hence, the general solution of (46) for the adiabatic bulk modulus is

$$A(p,\rho) = p\psi(\rho) \quad , \tag{50}$$

where $\psi(\rho)$ is an arbitrary function of the density. The ideal gas equation of state adiabatic bulk modulus (21) is a special case of (50) with

$$\psi(\rho) = \gamma \quad . \tag{51}$$

The Mie-Gruneisen equation of state adiabatic bulk modulus (23) follows from (50) by taking

$$\psi(\rho) = \Gamma + \frac{1}{\Gamma}\frac{\partial}{\partial \rho}(\rho\Gamma) \tag{52}$$

in terms of the Gruneisen coefficient. If we replace p by $p+p_e$ in (37), (40), (41), (43), (45), (46), (47) and (50), then the adiabatic bulk modulus for the stiff gas equation of state (25) is a special case of the modified (50), namely,

$$A(p,\rho) = (p+p_e)\psi(\rho) \quad , \tag{53}$$

obtained with $\psi(\rho)$ in (51) and

$$p_e = \frac{\rho_\infty a^2}{\gamma} \quad . \tag{54}$$

Accordingly, the hydrodynamics equations (31)-(33) can be reduced to a system of ordinary differential equations by introducing similarity variables obtained by integrating the characteristic equations (45). Similarity variables are invariant functions of the group generated by (40).

When $e_4 = 2e_3$, the characteristic equations (45) become

$$\frac{dr}{(e_2+e_3)r}=\frac{dt}{e_1+e_2 t}=\frac{du}{e_3 u}=\frac{dp}{2e_3 p}=\frac{d\rho}{0} \quad . \tag{55}$$

In (55), set $e_1=0$, $e_2=1$ and

$$1+e_3=\alpha \quad . \tag{56}$$

Then we have

$$\frac{dr}{\alpha r}=\frac{dt}{t}=\frac{du}{(\alpha-1)u}=\frac{dp}{2(\alpha-1)p}=\frac{d\rho}{0} \quad . \tag{57}$$

The solutions of the characteristic equations (57) are

$$\lambda=\frac{Cr}{t^{\alpha}}=\text{constant} \quad , \tag{58}$$

$$\frac{u}{t^{(\alpha-1)}}=\text{constant} \quad , \tag{59}$$

$$\frac{p}{t^{2(\alpha-1)}}=\text{constant} \quad , \tag{60}$$

and

$$\rho=\text{constant} \quad . \tag{61}$$

These invariant functions of the group generated by (40) comprise independent and dependent variables in terms of which the hydrodynamics equations (31)-(33) can be reduced to ordinary differential equations.

The invariant function λ in (58) contains only the original independent variables and is, therefore, taken as the independent variable for the reduction of the hydrodynamics equations (31)-(33) to ordinary differential equations. Let λ_s be the value of λ that corresponds to the position $R_s(t)$ of a shock. Then from (58) we have

$$C\,R_s(t)=\lambda_s t^{\alpha} \quad , \tag{62}$$

and the shock speed,

$$D(t)=\dot{R}_s(t) \quad , \tag{63}$$

is given by

$$C\,D(t)=\lambda_s \alpha\, t^{\alpha-1} \quad . \tag{64}$$

The invariant functions (59). (60), and (61) contain the original dependent variables of density, pressure, and velocity and are, therefore, taken as new dependent variables in terms of which the hydrodynamics equations (31)-(33) are reduced to ordinary differential equations. That is,

$$\rho = f(\lambda) \quad , \tag{65}$$

$$p = t^{2(\alpha-1)} g(\lambda) \quad , \tag{66}$$

and

$$u = t^{(\alpha-1)} h(\lambda) \quad , \tag{67}$$

where functions $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are determined by integrating the set of ordinary differential equations that come out of substituting (65)-(67) into the hydrodynamics equations (31)-(33).

In view of the shock speed result in (64), an alternative form of (65)-(67) that gives new nondimensional dependent variables and is useful for diverging and converging shock problems is

$$\rho = \rho_\infty f(\lambda) \quad , \tag{68}$$

$$p = \rho_\infty D^2(t) g(\lambda) \quad , \tag{69}$$

and

$$u = D(t) h(\lambda) \quad . \tag{70}$$

For some equations of state, for example, that of the stiff gas, the pressure on the left side of (69) is replaced with $p + p_e$, as in (53). Similarity variables for the Noh problem are special cases (65)-(67) with $\alpha = 1$. The shock speed for the Noh problem is from (64) a constant,

$$C\,D = \lambda_s \quad , \tag{71}$$

the new independent variable is given by (58) with $\alpha = 1$, and the new dependent variables can be taken as

$$\rho = f(\lambda) \quad , \tag{72}$$

$$p = g(\lambda) \quad , \tag{73}$$

and

$$u = h(\lambda) \quad , \tag{74}$$

from (65)-(67) with $\alpha = 1$.

4.1 SOLUTION OF THE NOH PROBLEM FOR THE IDEAL GAS EQUATION OF STATE

The hydrodynamics equations (31)-(33) with the adiabatic bulk modulus (21) for the ideal gas equation of state are

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{Nu}{r}\right) = 0 \quad , \tag{75}$$

$$\rho\left(u_t + u\,u_r\right) + p_r = 0 \quad , \tag{76}$$

and

$$p_t + up_r - \frac{\gamma p}{\rho}\left(\rho_t + u\,\rho_r\right) = 0 \quad . \tag{77}$$

By substituting the similarity variables (58) with $\alpha = 1$ and (72)-(74) into the hydrodynamics equations (75)-(77), we find the following set of ordinary differential equations:

$$\frac{df}{d\lambda} = \frac{\Delta_1}{\Delta} \quad , \tag{78}$$

$$\frac{dg}{d\lambda} = \frac{\Delta_2}{\Delta} \quad , \tag{79}$$

and

$$\frac{dh}{d\lambda} = \frac{\Delta_3}{\Delta} \quad , \tag{80}$$

where the definitions

$$\Delta = \gamma\, f\, g - f^2\left(h - \frac{\lambda}{C}\right)^2 \quad , \tag{81}$$

$$\Delta_1 = \frac{N}{\lambda} f^3 h\left(h - \frac{\lambda}{C}\right) \quad , \tag{82}$$

$$\Delta_2 = \frac{N\gamma}{\lambda} f^2 g h\left(h - \frac{\lambda}{C}\right) \quad , \tag{83}$$

and

$$\Delta_3 = -\frac{N\gamma}{\lambda} fgh \tag{84}$$

have been used. In the Noh problem, the region behind the shock lies in the domain $0 \le r \le R_s$, and the region ahead of the shock lies in the domain $R_s \le r < \infty$, and solutions of (78)-(80) are obtained subject to the Rankine-Hugoniot relations, a general form of which are

$$(u_1 - D)\rho_1 = (u_0 - D)\rho_0 = m \quad , \tag{85}$$

$$p_1 + mu_1 = p_0 + mu_0 \quad , \tag{86}$$

and

$$\varepsilon_1 + \frac{p_1}{\rho_1} + \frac{1}{2}(u_1 - D)^2 = \varepsilon_0 + \frac{p_0}{\rho_0} + \frac{1}{2}(u_0 - D)^2 \quad . \tag{87}$$

Here, quantities ahead of the shock are denoted with the subscript, 0, and those behind the shock, with the subscript, 1. In the case of the Noh problem it is assumed that

$$p_1 >>> p_0 \quad , \tag{88}$$

and

$$\varepsilon_1 + \frac{p_1}{\rho_1} >>> \varepsilon_0 + \frac{p_0}{\rho_0} \quad . \tag{89}$$

Also, for the Noh problem

$$u_1 = 0 \quad , \tag{90}$$

and the Rankine-Hugoniot relations (85)-(87) simplify to

$$\rho_1 = \left(\frac{D - u_0}{D}\right)\rho_0 \quad , \tag{91}$$

$$p_1 = (u_0 - D)u_0\, \rho_0 \quad , \tag{92}$$

and

$$\varepsilon_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_0(u_0 - 2D) \quad . \tag{93}$$

When the speed of the incoming material u_0 is specified, the simplified Rankine-Hugoniot relations (91)-(93) are three questions from which the shock speed D, the density ρ_1 behind the shock, and the pressure p_1 behind the shock are determined. The density ρ_0

just ahead of the shock in (91)-(93) is found by integrating the reduced hydrodynamics equations (78)-(80).

In the region ahead of the shock the assumption (88) is equivalent to setting $g=0$ in (78)-(80). Since all material ahead of the shock moves with speed u_0 in the Noh problem, $h(\lambda)=u_0$, equations (79) and (80) are satisfied identically. and the density distribution ahead of the shock is determined by integrating (78), which becomes

$$\lambda(\lambda-u_0C)\frac{df}{d\lambda}=Nu_0\,Cf \quad . \tag{94}$$

Integrating from the shock position λ_s out to infinity yields

$$\int_{f(\lambda_s)}^{f(\infty)}\frac{df}{f}=Nu_0C\int_{\lambda_s}^{\infty}d\lambda\frac{1}{\lambda(\lambda-u_0C)} \quad , \tag{95}$$

or

$$f(\lambda_s)=f(\infty)\left(\frac{\lambda_s-u_0C}{\lambda_s}\right)^N \quad . \tag{96}$$

Eliminating λ_s from (96) with (71), and noting that

$$f(\lambda_s)=\rho_0 \tag{97}$$

and

$$f(\infty)=\rho_\infty \quad , \tag{98}$$

we obtain

$$\rho_0=\rho_\infty\left(\frac{D-u_0}{D}\right)^N \quad , \tag{99}$$

where ρ_∞ is the normal density of the material. Also, by integrating (94) the density distribution ahead of the shock is found to be given by

$$\rho(r,t)=\rho_\infty\left(\frac{r-u_0t}{r}\right)^N=f(\lambda) \tag{100}$$

for $R_s(t)\le r\le\infty$, where

$$R_s(t)=tD \quad . \tag{101}$$

When the ideal gas equation of state,

$$p_1=(\gamma-1)\rho_1\varepsilon_1 \quad , \tag{102}$$

is substituted into the Noh problem form of the Rankine-Hugoniot relations (91)-(93), the energy balance (93) becomes

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}=\frac{1}{2}u_0\left(u_0-2D\right) \quad . \tag{103}$$

From (91) an (92) we have

$$\frac{p_1}{\rho_1}=-u_0 D \quad , \tag{104}$$

and combining (103) and (104) produces the shock speed

$$D=-\frac{1}{2}u_0(\gamma-1) \quad , \tag{105}$$

in which $u_0 < 0$. The compression ahead of the shock is found with (99) and (105) to be

$$\rho_0=\rho_\infty\left(\frac{\gamma+1}{\gamma-1}\right)^N \quad ; \tag{106}$$

Substituting (105) and (106) into (91) gives the compression behind the shock, namely,

$$\rho_1=\rho_\infty\left(\frac{\gamma+1}{\gamma-1}\right)^{N+1} \quad . \tag{107}$$

Also, by combining (105) and (106) with (92), we obtain the pressure behind the shock in the form,

$$p_1=\frac{\rho_\infty u_0^2(\gamma+1)^{N+1}}{2(\gamma-1)^N} \quad . \tag{108}$$

The results obtained in (105), (107), and (108) for the case of an ideal gas equation of state can be generalized to other equations of state whose adiabatic bulk modulus takes the form found in (50) or (53) because the corresponding hydrodynamics equations are invariant under the group of point transformations generated by (40).

4.2 SOLUTION OF THE NOH PROBLEM FOR A MIE-GRUNEISEN EQUATION OF STATE

It has been shown above that the hydrodynamics equations (31)-(33) are invariant in the case of the Mie-Gruneisen equation of state,

$$p=\rho\varepsilon\Gamma(\rho) \quad , \tag{109}$$

in which the Gruneisen coefficient $\Gamma(\rho)$ is that of (26). Hence, similarity variables of the type given in (72)-(74) and (58) with $\alpha = 1$ exist for this equation of state.

Let Γ_1 denote the value of the Gruneisen coefficient (26) evaluated at the compression behind the shock in the Noh problem, that is,

$$\Gamma_1 = \Gamma(\rho_1 / \rho_\infty) \quad . \tag{110}$$

Then the energy Rankine-Hugoniot relation (93) becomes

$$\left(1 + \frac{1}{\Gamma_1}\right)\frac{p_1}{\rho_1} = \frac{u_0(u_0 - 2D)}{2} \quad . \tag{111}$$

Substituting (104) into (111) and solving out for the shock speed yields

$$D = -\frac{u_0 \Gamma_1}{2} \quad . \tag{112}$$

With (112) the density behind the shock (91) becomes

$$\rho_1 = \frac{\Gamma_1 + 2}{\Gamma_1}\rho_0 \quad , \tag{113}$$

and the density ahead of the shock (99) reduces to

$$\rho_0 = \rho_\infty \left(\frac{\Gamma_1 + 2}{\Gamma_1}\right)^N \quad . \tag{114}$$

Combining (113) and (114) produces

$$\rho_1 = \rho_\infty \left(\frac{\Gamma_1 + 2}{\Gamma_1}\right)^{N+1} \quad , \tag{115}$$

which with (26) is a transcendental equation for the compression ρ_1 / ρ_∞ behind the shock. The shock speed is obtained by substituting the solution of (115) into (112), and the pressure behind the shock comes out of substituting (112) and (114) into (92) to obtain

$$p_1 = \frac{\rho_\infty u_0^2}{2} \frac{(\Gamma_1 + 2)^{N+1}}{\Gamma_1^N} \quad . \tag{116}$$

The Gruneisen coefficient that corresponds to the ideal gas equation of state is

$$\Gamma_1 = \gamma - 1 \quad . \tag{117}$$

If (117) is substituted into (112), (115) and (116), the results for the Mie-Gruneisen equation of state for the shock speed, compression behind the shock, and pressure behind the shock reduce to those found in (105), (108), and (108) for the ideal gas equation of state.

4.3 SOLUTION OF THE NOH PROBLEM FOR THE STIFF GAS EQUATION OF STATE

As demonstrated above, similarity solutions of the Noh problem exist for the stiff gas equation of state,

$$p = a^2(\rho - \rho_\infty) + (\gamma - 1)\rho\varepsilon \quad . \tag{118}$$

For this case the energy Rankine-Hugoniot relation (93) reduces to

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1} - \left(\frac{a^2}{\gamma-1}\right)\left(1-\frac{\rho_\infty}{\rho_1}\right) = \frac{u_0(u_0-2D)}{2} \quad . \tag{119}$$

The shock speed is determined in terms of the compression behind the shock by combining (104) and (119). The result is

$$D = -u_0\left[\frac{\gamma-1}{2} + \frac{a^2}{u_0^2}\left(1-\frac{\rho_\infty}{\rho_1}\right)\right] \quad . \tag{120}$$

A transcendental equation for the compression behind the shock that comes out of combining (91) (99), and (120) is

$$\frac{\rho_1}{\rho_\infty} = \left[\frac{\gamma+1+\dfrac{2a^2}{u_0^2}\left(1-\dfrac{\rho_\infty}{\rho_1}\right)}{\gamma-1+\dfrac{2a^2}{u_0^2}\left(1-\dfrac{\rho_\infty}{\rho_1}\right)}\right]^{N+1} \quad . \tag{121}$$

The pressure behind the shock is found with (92), (99), and (120) to be

$$p_1 = \frac{\rho_\infty u_0^2}{2}\frac{\left[\gamma+1+\dfrac{2a^2}{u_0^2}\left(1-\dfrac{\rho_\infty}{\rho_1}\right)\right]^{N+1}}{\left[\gamma-1+\dfrac{2a^2}{u_0^2}\left(1-\dfrac{\rho_\infty}{\rho_1}\right)\right]^{N}} \quad . \tag{122}$$

In the limit $a = 0$, the results in (120), (121), and (122) for the shock speed, the compression behind the shock, and the pressure behind the shock calculated with the stiff gas equation of state reduce to the ideal gas results given in (105), (107) and (108).

REFERENCES

1. W.F. Noh, "Errors for Calculations of Strong Shocks Using an Artificial Viscosity and an Artificial Heat Flux," J. Comp. Phys., 72, 78-120 (1987).

2. S. I. Anismov and V. A. Kravchenko, "Shock Wave in Condensed Matter Generated by Impulsive Load," Z. Naturforsch., **40a**, 8-13 (1985).

This report has been reproduced directly from the best available copy.

It is available to DOE and DOE contractors from
the Office of Scientific and Technical Information,
P.O. Box 62,
Oak Ridge, TN 37831.
Prices are available from
(615) 576-8401.

It is available to the public from the
National Technical Information Service,
US Department of Commerce,
5285 Port Royal Rd.,
Springfield, VA 22616.

Los Alamos
NATIONAL LABORATORY
Los Alamos, New Mexico 87545