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TITLE GROUP-INVARIANT SOLUTIONS OF HYDRODYNAMICS AND RADIATION HYDRODYNAMICS

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GROUP-INVARIANT SOLUTIONS OF HYDRODYNAMICS AND RADIATION HYDRODYNAMICS

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Using the property of invariance under Lie groups of transformations, the equations of hydrodynamics are transformed from partial differential equations to ordinary differential equations, for which special analytic solutions can be found. These particular solutions can be used for (1) numerical benchmarks, (2) the basis for analytic models, and (3) insight into more general solutions. Additionally, group transformations can be used to construct new solutions from existing ones. A space-time projective group is used to generate complicated solutions from simpler solutions. Discussion of these procedures is presented along with examples of analytic solutions of 1, 2 and 3-D hydrodynamics.

1. Introduction

The construction and use of large-scale hydrodynamics codes is an integral part of the efforts at many organizations internationally. A fundamental requirement for such codes is to adequately calculate simple test problems for which the exact solution is known. Such benchmark problems are therefore a basic need for any groups developing nontrivial numerical codes.

Since many processes in which organizations are interested require at least the solution of hydrodynamics equations, nontrivial benchmark problems to such equations are important. In the past, many 1-D test problems have been used, but there has been a lack of multidimensional (2 and 3-D) nontrivial analytic solutions. Often, a 1-D problem has been run in a 2-D mode, which does not test a wide variety of physical/numerical processes. Presented here are a number of 2 and 3-D analytic solutions to the hydrodynamic equations that may immediately be used as benchmarks. The solutions were found using Lie group methods for the reduction of partial differential equations; details of this method can be found in the texts listed in Reference 1.

The use of Lie groups to simplify/solve differential equations has enjoyed renewed popularity in the last few decades. By identifying the continuous transformation groups under which the equations remain invariant, new sets of coordinates can be identified under which the equations become simpler. Ordinary differential equations (ODE's) undergo a reduction of order and partial differential equations (PDE's) experience a reduction in the number of independent variables until they become ODE's.

2. Model

In this work the equations for one-temperature inviscous hydrodynamics are considered. A restriction to a perfect gas equation of state is made, so the material pressure and energy are written as

$$P = \Gamma\rho T, \quad E = \frac{\Gamma}{\gamma - 1}T,$$

where Γ is the gas constant, γ is the adiabatic exponent and T is the material temperature. (Use of a more general equation of state, including a power law form, for the 1-D case can be found in Reference 2.) Heat conduction is included in the diffusion approximation, where the heat flux F is represented through a nonlinear Fourier's Law, $F = -\kappa(\rho, T)\nabla T$. A general energy source term $S(\mathbf{x}, t, \rho, \mathbf{u}, T)$ is also included.

With these assumptions, the equations of mass, momentum, and energy conservation become

$$\begin{aligned} &\rho_t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0, \\ &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla(\Gamma\rho T) = 0, \\ &T_t + \mathbf{u}\cdot\nabla T + (\gamma - 1)T\nabla\cdot\mathbf{u} \\ &\qquad - \frac{\gamma - 1}{\Gamma\rho}\nabla\cdot\kappa\nabla T - \frac{\gamma - 1}{\Gamma\rho}S = 0 \end{aligned} \tag{1}$$

The 3-dimensional velocity is written as $\mathbf{u} = (u, v, w)$.

We look for continuous transformations which leave these equations invariant by introducing the differential operator

$$U = \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^z \frac{\partial}{\partial z} + \xi^t \frac{\partial}{\partial t} + \eta^\rho \frac{\partial}{\partial \rho} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta^w \frac{\partial}{\partial w} + \eta^T \frac{\partial}{\partial T},$$

where each of the functions ξ^k and η^k depend only on the independent variables x, y, z, t, and the dependent variables ρ, u, v, w, and T. This operator is extended to action on the derivatives in (1) through prolongation formulas described in all the texts listed in Reference 1. Invariance of these equations is invoked by demanding that U operating on these equations gives back some arbitrary function times the original equations, which says that the equations are invariant on the solution manifold. These invariance conditions provide relations to determine the unknown functions ξ^k and the η^k.

The Lie groups of point transformations under which Equations (1) remain invariant are generated by the operators

$$\begin{aligned}
U_x &= \frac{\partial}{\partial x} \equiv \partial_x, \\
U_y &= \partial_y, \\
U_z &= \partial_z, \\
U_t &= \partial_t, \\
U_{Gx} &= t\partial_x + \partial_u, \\
U_{Gy} &= t\partial_y + \partial_v, \\
U_{Gz} &= t\partial_z + \partial_w, \\
U_{xy} &= -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \\
U_{yz} &= -z\partial_y + y\partial_z - w\partial_v + v\partial_w, \\
U_{zx} &= -x\partial_z + z\partial_x - w\partial_u + u\partial_w, \\
U_{st} &= t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2T\partial_T, \\
U_{ss} &= x\partial_x + y\partial_y + z\partial_z - q\rho\partial_\rho \\
&\quad + u\partial_u + v\partial_v + w\partial_w + 2T\partial_T, \\
U_{s\rho} &= \rho\partial_\rho, \\
U_p &= xt\partial_x + yt\partial_y + zt\partial_z + t^2\partial_t - c\rho t\partial_\rho \\
&\quad + (x - ut)\partial_u + (y - vt)\partial_v + (z - wt)\partial_w - 2Tt\partial_T,
\end{aligned}$$

with the following conditions for the conductivity κ and energy source S:

$$\begin{aligned}
\rho\kappa_\rho(a_{s\rho} - qa_{ss} - qta_p) + 2T\kappa_T(a_{ss} - a_{st} - ta_p) \\
= a_{s\rho} - a_{st} + (2 - q)a_{ss} - qa_p t, \\
US - S(2a_{ss} - 3a_{st} - 4ta_p) \\
- qa_p \frac{\Gamma T}{\gamma - 1}\left(\gamma - \frac{q+2}{q}\right) = 0.
\end{aligned} \qquad (2)$$

The complete invariance operator is a linear combination of the separate operators,

$$U = a_x U_x + a_y U_y + \ldots + a_{s\rho} U_{s\rho} + a_p U_p,$$

where the a_i's are arbitrary constants.

These groups represent space translations (U_x, U_y, U_z), time translation (U_t), Galilean boosts (U_{Gx}, U_{Gy}, U_{Gz}), rotations (U_{xy}, U_{yz}, U_{zx}), space scaling (U_{ss}), time scaling (U_{st}), density scaling ($U_{s\rho}$), and a space-time projective group (U_p).

For the remainder of this work both heat conduction and the energy source S are neglected. The conditions for the inclusion of these terms are given by (2), and several 1-D analytic solutions including heat conduction are given in Reference 4. There is no problem finding similarity solutions to multidimensional hydrodynamics with either conduction or the source term retained. It is, however, much more difficult to solve analytically the reduced ordinary differential equations with these terms included. Numerical solutions of these ODE's are straightforward.

3. Reduction to ODE's

Consider an example PDE $H(x_i, y_i, \ldots) = 0$ with n independent variables $\mathbf{x}$ and m dependent variables $\mathbf{y}$ which is invariant under a differential operator $U = \sum_i \xi_i \partial_{x_i} + \sum_i \eta_i \partial_{y_i}$. That is, $UH = 0$ whenever $H = 0$. Since U is a linear differential operator, we know (by the method of characteristics) we can rewrite $H = 0$ in terms of the $n - 1 + m$ integration constants c_i of the characteristic equations

$$\frac{dx_1}{\xi_1} = \ldots = \frac{dx_n}{\xi_n} = \frac{dy_1}{\eta_1} = \ldots = \frac{dy_m}{\eta_m}.$$

Thus $H = 0$ becomes $G(c_1, \ldots, c_{n-1+m}) = 0$, and the number of independent variables has been reduced by one.

The use of the integration constants from the invariance characteristic equations as new variables is seen as the identification and introduction of a new "preferred" coordinate system indicated by the symmetries of the differential equations. For PDE's with two independent variables a single such transformation reduces the equations to ODE's. To reduce PDE's with n independent variables to ODE's, $n-1$ such reductions must be made. These multiple reductions are facilitated by examining the structure of the associated Lie algebra. Specifically, after we perform the first of several reductions, we wish the resulting equations to retain the symmetries of the groups used in the first step so that the process can be continued. This is guaranteed through the following theorem: If a differential equation Δ is invariant under a Lie group G with a *normal subgroup* $S \subset G$, then the reduced equation Δ/S obtained using the group S will be invariant under the *quotient group* G/S. Complete description as well as examples of this procedure can be found in Reference 3.

To illustrate this procedure we consider the two-dimensional axisymmetric simplification of Equations (1). We work in spherical coordinates with independent variables r, θ, and t, and let u and v be the r- and θ components of the velocity, respectively. We choose two subgroups allowed in this geometry for a double reduction, using first $U^1 = U_p + \alpha U_{tp}$ and then $U^2 = U_s + \beta U_{sp}$, where α and β are arbitrary constants, for the two reductions. We identify $U^1 = rt\partial_r + t^2\partial_t + \rho(\alpha - 3t)\partial_\rho + (r - ut)\partial_u - vt\partial_v - 2Tt\partial_T$ and $U^2 = r\partial_r + \rho(\beta - 3)\partial_\rho + u\partial_u + v\partial_v + 2T\partial_T$. We do not need to calculate the differential equations obtained with the first reduction, we can proceed to find the invariants of both U^1 and U^2 and do the double reduction in one step.

The characteristic equations for U^1 are

$$\frac{dr}{rt} = \frac{d\theta}{0} = \frac{dt}{t^2} = \frac{d\rho}{\rho(\alpha - 3t)} = \frac{du}{r - ut} = \frac{dv}{-vt} = \frac{dT}{-2Tt}$$

The integration constants of these equations are the new variables

$$c_1 = \frac{r}{t},\ c_2 = \theta,\ c_3 = \rho t^3 e^{\alpha/t},\ c_4 = ut - r,$$
$$c_5 = vt,\ c_6 = Tt^2$$

Since U^1 is a normal subgroup of U^2, we are assured that the equations obtained by reducing with U^1 will inherit the property of invariance under U^2, and a further reduction is then possible. We can therefore write U^2 in terms of these new variables as $U^2 = f_1\partial_{c_1} + \ldots + f_6\partial_{c_6}$, where the functions f_i can be calculated from $f_i = U^2 c_i$. We find $U^2 = c_1\partial_{c_1} + c_3(\beta - 3)\partial_{c_3} + c_4\partial_{c_4} + c_5\partial_{c_5} + 2c_6\partial_{c_6}$, with the characteristic equations

$$\frac{dc_1}{c_1} = \frac{dc_2}{0} = \frac{dc_3}{c_3(\beta - 3)} = \frac{dc_4}{c_4} = \frac{dc_5}{c_5} = \frac{dc_6}{2c_6}.$$

The integration constants of this set of equations become the new similarity variables:

$$\lambda = \theta, H(\lambda) = \rho r^{3-\beta} t^{\beta} e^{\alpha/t}, U(\lambda) = \frac{ut^2}{r} - t,$$
$$V(\lambda) = \frac{vt^2}{r}, G(\lambda) = T\frac{t^4}{r^2}.$$

The final step is to transform the original PDE's into ODE's in these new variables. Using the chain rule, we calculate the derivatives required in (1) in terms of the new variables. E.g.,

$$\rho_t = \frac{\partial}{\partial t}\left[H(\lambda) r^{\beta-3} t^{-\beta} e^{-\alpha/t}\right]$$
$$= r^{\beta-3} e^{-\alpha/t} t^{-\beta}\left[H\frac{\alpha}{t^2} - H\frac{\beta}{t} + H'\lambda_t\right].$$

For this case $\lambda_t = 0$ since $\lambda = \theta$. When these expressions are substituted into (1), the equations become ODE's for $H(\lambda)$, $U(\lambda)$, $V(\lambda)$ and $G(\lambda)$. Any solution of these ODE's provides a particular solution to (1).

4. Analytic Solutions

In this section are listed a number of analytic solutions to the ordinary differential equations found by using various similarity reductions described in Reference 3. That work listed a minimal and complete set of similarity reductions for 2-D axisymmetric geometry, taking the partial differential equations into ordinary differential equations. The present Solutions 1 - 7 are specific solutions related to this previous work, and their relationship and method of solution of the reduced ordinary differential equations are listed here.

Each analytic solution gives the material properties (density, velocities, temperature) as a function of space and time. Unless otherwise noted, the velocities are always $(u, v, w) \equiv (u^r, u^\theta, u^\phi)$, being radial, theta and phi velocities in standard spherical coordinates. The variable r is the spherical radius and R is the cylindrical radius.

2-D axisymmetric flow, (r, θ) or (R, z):

Solutions 1 - 4 come from $\mathcal{H}_7$ with the ansatz $U = U_0(a + b\cos^2\theta)$, $V = V_0\sin\theta\cos\theta$.

1. $\beta \neq -2$, $bU_0 + V_0 = 0$, $aU_0 = 0$:

$$\rho(r,\theta,t) = \rho_0 t^{-2(\beta+1)/(\gamma+1)}(r\cos\theta)^\beta$$
$$u(r,\theta,t) = \frac{2}{(\gamma+1)}\frac{r}{t}\cos^2\theta$$
$$v(r,\theta,t) = -\frac{2}{(\gamma+1)}\frac{r}{t}\sin\theta\cos\theta$$
$$T(r,\theta,t) = \frac{2(\gamma-1)}{\Gamma(\gamma+1)^2(\beta+2)}\left(\frac{r}{t}\right)^2\cos^2\theta$$

2. $\beta \neq -2$, $bU_0 + V_0 = 0$, $aU_0 = 1$:

$$\rho(r,\theta,t) = \rho_0 t^{2-3[3(\beta+1)\gamma-1]/(\gamma+1)}(r\cos\theta)^\beta$$
$$u(r,\theta,t) = \frac{r}{t}\left(1 - 3\frac{\gamma-1}{\gamma+1}\cos^2\theta\right)$$
$$v(r,\theta,t) = 3\frac{\gamma-1}{\gamma+1}\frac{r}{t}\sin\theta\cos\theta$$
$$T(r,\theta,t) = \frac{6(\gamma-1)(2-\gamma)}{\Gamma(\gamma+1)^2(\beta+2)}\left(\frac{r}{t}\right)^2\cos^2\theta$$

3. $\beta = -2$, $aU_0 = 0$:

$$\rho(r,\theta,t) = \rho_0 t^{2[illegible]} r^{-2}(\sin\theta)^{[illegible]}(\cos\theta)^{[illegible]}$$
$$u(r,\theta,t) = \frac{r}{t}\cos^2\theta$$
$$v(r,\theta,t) = -\frac{r}{t}\sin\theta\cos\theta$$
$$T(r,\theta,t) = T_0\left(\frac{r}{t}\right)^2(\sin\theta)^{[illegible]}(\cos\theta)^{[illegible]}$$

4. $\beta = -2$, $aU_0 = 1$:

$$\rho(r,\theta,t) = \rho_0 t^{2(1-\gamma)} r^{-2}(\sin\theta)^{2\gamma-4}(\cos\theta)^{2-2\gamma}$$
$$u(r,\theta,t) = \frac{r}{t}(1 - \cos^2\theta)$$
$$v(r,\theta,t) = \frac{r}{t}\sin\theta\cos\theta$$
$$T(r,\theta,t) = T_0\left(\frac{r}{t}\right)^2(\sin\theta)^{4-2\gamma}(\cos\theta)^{2\gamma-2}$$

5. $\mathcal{H}_{12}$ with the ansatz $A = 0$:

$$\rho(R,z,t) = \rho_0\exp\left(\alpha t + \beta z - a\beta t^2\right)$$
$$u^R(R,z,t) = 0,$$
$$u^z(R,z,t) = -\frac{\alpha}{\beta} + 2at$$
$$T(R,z,t) = -\frac{2a}{\Gamma\beta}$$

6. $\mathcal{H}_{14}$ with the ansatz $U = 0$:

$$\rho(r,\theta,t) = \rho_0 r^{2[1-2a-\gamma(a+1)]/(2-\gamma)}(\sin\theta)^{-2/(\gamma+1)}$$
$$u(r,\theta,t) = 0$$
$$v(r,\theta,t) = v_0 r^a(\sin\theta)^{(1-\gamma)/(1+\gamma)}$$
$$T(r,\theta,t) = v_0^2\frac{1-\gamma}{2\Gamma\gamma}r^{2a}(\sin\theta)^{(2-2\gamma)/(1+\gamma)}$$

7. $\mathcal{H}_{14}$ with the ansatz $U = U_0\cos\theta$, $V = V_0\sin\theta$

$$\rho(r,\theta,t) = \rho_0 r^{-2\gamma/(2\gamma-1)}(\sin\theta)^{(2-2\gamma)/(2\gamma-1)}$$
$$u(r,\theta,t) = u_0\cos\theta$$
$$v(r,\theta,t) = u_0\frac{1-\gamma}{\gamma}\sin\theta$$
$$T(r,\theta,t) = u_0^2\frac{(\gamma-1)(2\gamma-1)}{2\Gamma\gamma^3}\sin^2\theta$$

Next are presented analytic solutions in geometries other than axisymmetric. First is 2-D cylindrical geometry in a plane with no z dependence. Following that is a 3-D solution.

2-D cylindrical solutions (R, ϕ)

Here the groups ($U_{[illegible]} + c_1 U_{[illegible]} + c_2 U_{[illegible]} + c_3 U_{[illegible]}$, $U_{[illegible]} + c_4 U_{[illegible]} + c_5 U_{[illegible]}$) were used to generate the similarity variables

$$\lambda = R^a t^b e^{-c\phi},\ \rho = H(\lambda)\left(\frac{R}{t}\right)^d t^e,\ u^R = U(\lambda)\frac{R}{t},$$

$$u^\phi = V(\lambda)\frac{R}{t},\ T = G(\lambda)\left(\frac{R}{t}\right)^2.$$

8. $\gamma = 2,\ e = 0$:

$$\rho(R,\phi,t) = \rho_0\left(\frac{R}{t}\right)^2$$

$$u^R(R,\phi,t) = \frac{R}{2t}$$

$$u^\phi(R,\phi,t) = v_0\frac{R}{t}$$

$$T(R,\phi,t) = \frac{4v_0^2+1}{16\Gamma}\left(\frac{R}{t}\right)^2$$

9. $\gamma = 2,\ e = 0$:

$$\rho(R,\phi,t) = \rho_0 R^{-2}\left(\frac{R}{t}\right)^a$$

$$u^R(R,\phi,t) = \frac{R}{t}$$

$$u^\phi(R,\phi,t) = v_0 R^{-1}\left(\frac{R}{t}\right)^b$$

$$T(R,\phi,t) = \frac{v_0^2}{\Gamma(a+2b-4)}R^{-2}\left(\frac{R}{t}\right)^{2b}$$

10.

$$\rho(R,\phi,t) = \rho_0\left(\frac{R}{t}\right)^{d-bf} R^{f(a+b)}e^{-c\phi}$$

$$u^R(R,\phi,t) = \frac{R}{\gamma t}$$

$$u^\phi(R,\phi,t) = \frac{-d+(2+d+af)/\gamma}{c}\frac{R}{t}$$

$$T(R,\phi,t) = \frac{(2-\gamma)(1-\gamma)R^2}{\Gamma\gamma^2[(\gamma-1)(af+2)+fb\gamma-d]t^2}$$

with $c^2 = \frac{\gamma}{1-\gamma}[f(b+a/\gamma)+(d+2)/\gamma-d]$

$$\times[(\gamma-1)(af+2)+fb\gamma-d]$$

11. $V = 0$

$$\rho(R,\phi,t) = \rho_0\left(\frac{R}{t}\right)^a t^{-2}e^{c\phi}$$

$$u^R(R,\phi,t) = \frac{R}{t}$$

$$u^\phi(R,\phi,t) = 0$$

$$T(R,\phi,t) = T_0\left(\frac{R}{t}\right)^a t^{2(1-a)}e^{-c\phi}$$

3-D solution, spherical coordinates (r,θ,ϕ): For the 3-D reduction to ordinary differential equations, the groups $\langle U_{st}+c_1U_{ss}+c_2U_{xy}+c_3U_{s\rho}, U_{st}+c_4U_{ss}+c_5U_{s\rho}, U_{st}+c_6U_{s\rho}\rangle$ were used, which generated the similarity variables

$$\lambda = \theta,\ \rho = H(\lambda)t^d r^b e^{c\phi},\ u^r = U(\lambda)\frac{r}{t},$$

$$u^\theta = V(\lambda)\frac{r}{t},\ u^\phi = W(\lambda)\frac{r}{t},\ T = G(\lambda)\frac{r^2}{t^2}.$$

12. $V = 0,\ H = H_0 + H_1(\sin\theta)^a,\ c = 0,\ \gamma = 5/3$:

$$\rho(r,\theta,\phi,t) = t^{-(b+3)/2}r^b(\rho_0+\rho_1\sin^a\theta)$$

$$u(r,\theta,\phi,t) = \frac{r}{2t}$$

$$v(r,\theta,\phi,t) = 0$$

$$w(r,\theta,\phi,t) = \pm\frac{r}{4t}\left[\frac{4\Gamma T_0(b+2)(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2-a)} + \frac{a}{(b+2-a)}\right]^{1/2}$$

$$T(r,\theta,\phi,t) = \left(\frac{r}{4\Gamma t}\right)^2\left[\frac{4\Gamma T_0(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2)(b+2-a)} + \frac{1}{(b+2-a)}\right]$$

Figure 1 shows the material trajectories for Solution 12 with the choices $a = 1$ and $b = 2$.

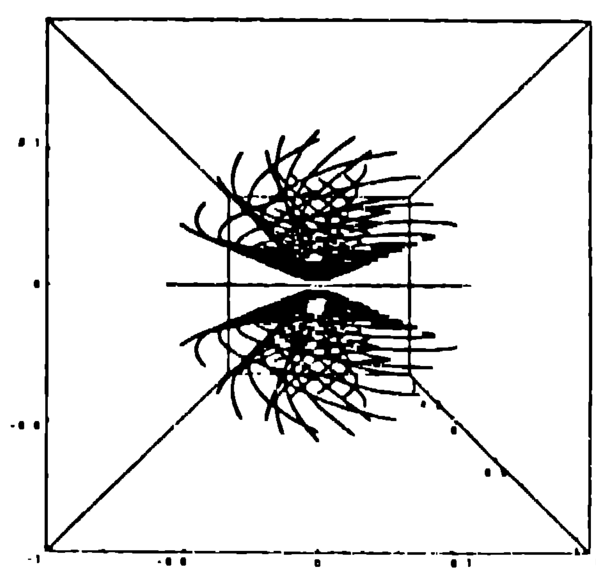

Figure 1 Material trajectories for Solution 12 with $\rho_0 = 0,\ \rho_1 = 1,\ a = 1,\ b = 2$

Another combination of groups $(U_t, U_{s1}+c_1U_{s\cdot} + c_2U_{xy}+c_3U_{s\rho}, U_{s\cdot}+c_4U_{s1}+c_5U_{s\rho})$ produces the similarity variables

$$\lambda = \theta,\ \rho = H(\lambda)r^a e^{c\phi},\ u^r = U(\lambda)r^b e^{d\phi},$$
$$u^\theta = V(\lambda)r^b e^{d\phi},\ u^\phi = W(\lambda)r^b e^{d\phi},\ T = G(\lambda)r^{2b}e^{2d\phi}.$$

The use of time translation alone in one generator creates a steady-state solution.

5. Extension of Solutions

Since the considered Lie groups of point transformations are invariance transformations, they transform solutions into other solutions. Given any solution of (1), we can use the global group transformations[3] to construct a further set of solutions to (1) with various choices of the group parameters. For the axisymmetric case, the global transformations for the allowed transformations are

$$\begin{aligned}
\bar{r} &= \frac{e_3 r}{1 - st},\\
\bar{\theta} &= \theta,\\
\bar{t} &= \frac{e_5 t}{1 - st} + \tau,\\
\bar{\rho} &= e_7 e_3^{-3}\rho(1 - st)^3,\\
\bar{u} &= e_3 e_5^{-1}[u(1 - st) + sr],\\
\bar{v} &= e_3 e_5^{-1} v(1 - st),\\
\bar{T} &= e_3^2 e_5^{-2} T(1 - st)^2,
\end{aligned} \tag{3}$$

with arbitrary constants e_i, s, and τ. Therefore, if $\Psi(r,\theta,t)$ is a solution to (1), then so also is $\bar{\Psi}(\bar{r},\bar{\theta},\bar{t})$.

All these additional free parameters are trivial except for s, which is the projective group parameter. The inclusion of s must be consistent with the conditions given in (2) (where s was called a_p), which for Solutions 1 – 7 (2-D axisymmetric, no conduction or source) is $\gamma = 5/3$. This projective transformation can generate a nontrivial extension of an existing solution. It is useful to examine the effect of this projective transformation on simple 1-D solutions.

We consider a 1-D flow where the velocity is $u = u_0 r/t$, which gives $r = r_0|t|^{u_0}$ for the trajectories of material flow. Many analytic 1-D solutions of this form can be found, for example, in Reference 4. Under the action of this projective group these relations are transformed into

$$u = \left(\frac{r}{t}\right)\frac{u_0 + st}{1 + st},\quad r = r_0|t|^{u_0}|1 + st|^{1-u_0}.$$

The value $u_0 = 0$ ($u = 0$) is a trivial solution to (1), and under the action of this projective group becomes a nontrivial solution. Figure 2 shows the effect of the projective group on this solution. For $u_0 = 1$ this projective transformation is an identity transformation, so solutions with trajectories shown in Figure 2b are unchanged. An interesting 1-D solution, Solution 2 from Reference 4, has the value $u_0 = 1$, $3/4$, or $1/2$ for planar, cylindrical, or spherical geometries, respectively, for a $\gamma = 5/3$ material. For $u_0 = 1/2$, the material trajectories are shown in Figure 3a and the trajectories for the corresponding projected solution are shown in Figure 3b. As in Figures 2a and 2b, we see

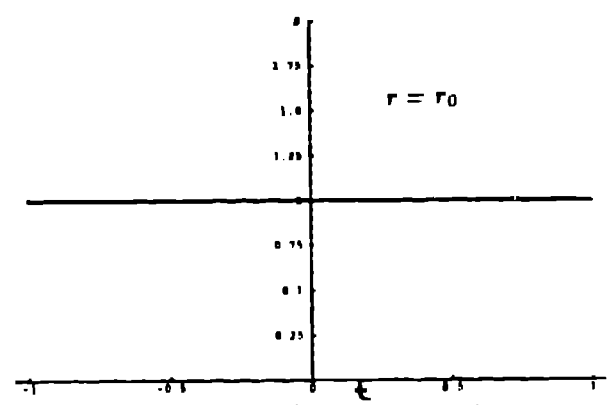

Figure 2a. Material trajectory with $u_0 = 0$.

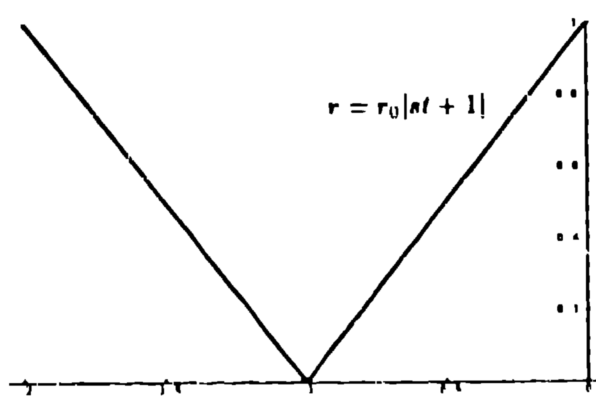

Figure 2b. Material trajectory after projective transformation

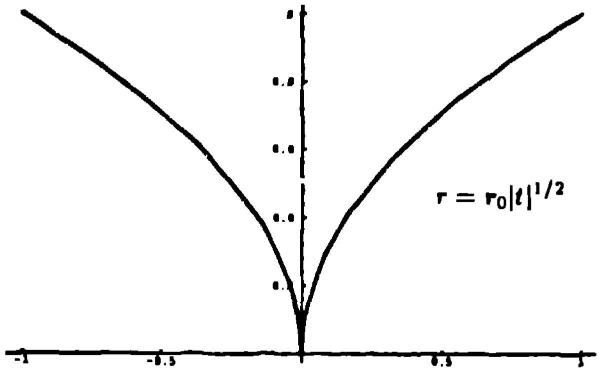

Figure 3a. Material trajectory for $u_0 = 1/2$.

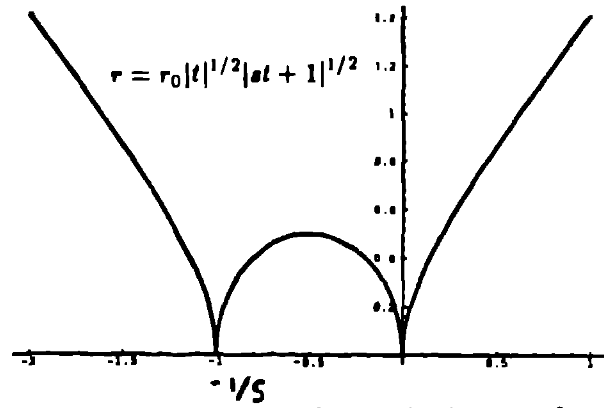

Figure 3b. Trajectory after projective transformation. Note the region bounded in space.

that the projective group has introduced another zero on the time axis. In Figure 3b we find a bounded solution between times $t = -1/s$ and $t = 0$. This solution begins with a point explosion (like a "Big Bang"), expands, turns around, and collapses back into a point.

The transformations (3) can also be used in 2 and 3 dimensions to generate new solutions from old ones. For example, Solution 2 becomes, under the extension (3),

13. (2-D axisymmetric)

$$\rho(r,\theta,t) = \rho_0(r\cos\theta)^\beta(t-\tau)^{-\beta-3+3(\beta+1)(\gamma-1)/(\gamma+1)} \times [c_5 + s(t-\tau)]^{-3(\beta+1)(\gamma-1)/(\gamma+1)}$$

$$u(r,\theta,t) = \frac{c_5 r}{s\left\{[t-\tau+c_5/(2s)]^2 - c_5^2/(4s^2)\right\}}\left(1 - 3\frac{\gamma-1}{\gamma+1}\cos^2\theta\right) + \frac{sr}{c_5 + s(t-\tau)}$$

$$v(r,\theta,t) = \frac{3c_5(\gamma-1)r\sin\theta\cos\theta}{s(\gamma+1)\left\{[t-\tau+c_5/(2s)]^2 - \frac{c_5^2}{4s^2}\right\}}$$

$$T(r,\theta,t) = c_5^2\frac{6(\gamma-1)(2-\gamma)}{\Gamma(\gamma+1)^2(\beta+2)}\cos^2\theta \times \left[\frac{r}{s\left\{[t-\tau+c_5/(2s)]^2 - c_5^2/(4s^2)\right\}}\right]^2,$$

with $s(\gamma - 5/3) = 0$ (Note, c_5 is not necessary here. We could let $s = s/c_5$, and c_5 is replaced by 1.)

The 3-D solution can also be extended in the same manner using time translation (τ) and the projective group (s). This solution becomes

14. (3-D, $\gamma = 5/3$)

$$\rho(r,\theta,\phi,t) = [(t-\tau)(1+s(t-\tau))]^{-(b+3)/2} \times r^b(\rho_0 + \rho_1\sin^a\theta)$$

$$u(r,\theta,\phi,t) = \frac{r}{2s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}} + \frac{sr}{1+s(t-\tau)}$$

$$v(r,\theta,\phi,t) = 0$$

$$w(r,\theta,\phi,t) = \pm\frac{1}{4}\left[\frac{4\Gamma T_0(b+2)(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2-a)} + \frac{a}{b+2-a}\right]^{1/2} \times \frac{r}{s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}}$$

$$T(r,\theta,\phi,t) = \frac{1}{4\Gamma}\left[\frac{4\Gamma T_0(\sin\theta)^{b+2}}{\rho_0+\rho_1\sin^a\theta} - \frac{a\rho_0}{(\rho_0+\rho_1\sin^a\theta)(b+2)(b+2-a)} + \frac{1}{(b+2-a)}\right] \times \left[\frac{r}{s\{[t-\tau+1/(2s)]^2 - 1/(4s^2)\}}\right]^2.$$

The projective group has introduced another zero on the plot of material trajectories, just as Figure 3a went into Figure 3b. This solution therefore has a bounded portion which involves a point explosion that spins outward, stops expanding and then collapses back onto itself as it continues to spin.

6. Boundary conditions

The analytic solutions as given above contain no boundary conditions, which must be taken into account for numerical solutions. There are two approaches to this concern. The first and simplest is to consider a finite region initialized with the properties of any of the analytic solutions with no consideration of special boundary conditions. In this approach the evolution at the boundary immediately deviates from the analytic solution, and a rarefaction wave propagates into the material of interest. The solution is valid only in the region which has not felt this rarefaction wave. This approach, while simplest to implement, causes the region of validity to shrink as the problem evolves.

The second approach is to apply the correct boundary conditions at the edge of the problem. This is immediate if the calculation is Eulerian. For a Lagrangian calculation one must calculate the location of the boundary at all times and the appropriate material properties for that location must be imposed. One concern with this approach is that small errors made in these boundary conditions can propagate into the problem and confuse the investigation of internally generated errors. For this reason we generally use the first and simplest approach. More general discussions of the treatment of boundary conditions can be found in References 2 and 4.

7. Summary

In this work we investigated the Lie group invariance properties of the 3-D hydrodynamics equations, including nonlinear conduction and an arbitrary energy source. Using these properties we constructed preferred coordinate systems in which the PDE's are transformed into ODE's.

This procedure is therefore a deterministic method for constructing similarity solutions, and includes dimensional analysis as a subset. The reduced ODE's have been solved for a few cases to provide analytic solutions to multidimensional hydrodynamics. These solutions can be used as numerical benchmarks for hydro codes.

We also demonstrated the property of transformation of solutions into new solutions using the global transformations of the allowed Lie groups. In particular, the use of the projective transformation generates nontrivial solutions from trivial ones, and can also provide quite complicated time-dependent solutions such as the given 3-D spinning/expansion/collapse solution, Solution 14.

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