LA-4992

c.3

A Geometric Interpretation of the Dirac Equation

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.95

LA-4992
UC-34
ISSUED: August 1972

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

A Geometric Interpretation of the Dirac Equation

by

R. Lee Aamodt

A GEOMETRIC INTERPRETATION OF THE DIRAC EQUATION

by

R. Lee Aamodt

ABSTRACT

In the space of special relativity, a Lorentz transformation and a rotation of the spatial axes through an imaginary angle generate two new coordinate systems in which a vector is described. The Dirac equation appears as a relation between the sums and differences of the components of the vector in the new coordinate systems.

The Dirac equation for a free particle may be written as two equations which resemble spinor expressions for rotations:[1]

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c/\left(E-m_0c^2\right)\begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \quad (1)$$

and

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = c/\left(E+m_0c^2\right)\begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (2)$$

It is not difficult to show that Eqs. (1) and (2) arise from two rotations of a four-dimensional Cartesian coordinate system through complex angles. One of these rotations is a Lorentz transformation and the other is a complex rotation in which only the spatial axes are transformed.

The derivation is simpler if the angles of rotation are at first assumed to be real so that ordinary spinor expressions for rotations can be used.

Consider a rotation in ordinary space through the angle $2\cot^{-1}\alpha$. Let the direction cosines of the axis of rotation be α_j $(j = 1,2,3)$.

Define two spinor components,

$$\xi_0 = \left(\alpha-i\alpha_3\right)/\left(1+\alpha^2\right)^{1/2} \quad (3a)$$

$$\xi_1 = \left(\alpha_2-i\alpha_1\right)/\left(1+\alpha^2\right)^{1/2} \quad (3b)$$

and the array

$$\xi = \begin{pmatrix} \xi_0 & -\xi_1^* \\ \xi_1 & \xi_0^* \end{pmatrix}. \quad (4)$$

The adjoint of ξ may be written $\xi^\dagger$, and we note that

$$\xi\xi^\dagger = \xi^\dagger\xi = 1 \quad . \quad (5)$$

A vector with components x'_j $(j = 1,2,3)$ before the rotation will have components x_j afterward. The x_j may be written in the form

$$\begin{pmatrix} x_3 & x_1-ix_2 \\ x_1+ix_2 & -x_3 \end{pmatrix} = \xi^\dagger\begin{pmatrix} x'_3 & x'_1-ix'_2 \\ x'_1+ix'_2 & -x'_3 \end{pmatrix}\xi, \quad (6)$$

as is easily verified.[2]

A fourth component of x which is unchanged by the rotation may be added to Eq. (6) in the form

$$ix_4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = ix'_4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \xi^\dagger\begin{pmatrix}ix'_4 & 0\\ 0 & ix'_4\end{pmatrix}\xi \quad . \qquad (7)$$

Let

$$M = \begin{pmatrix}x_3+ix_4 & x_1-ix_2\\ x_1+ix_2 & -x_3+ix_4\end{pmatrix} \quad . \qquad (8)$$

The spatial rotation of Eq. (6) may then be written as

$$M = \xi^\dagger M'\xi \quad . \qquad (9)$$

A different rotation, involving all four dimensions, may be written as

$$M'' = \xi^\dagger M\xi^\dagger \quad . \qquad (10)$$

Then from Eqs. (9) and (10),

$$M'' = \left(\xi^\dagger\right)^2 M'$$

or

$$\xi M'' = \xi^\dagger M' \quad . \qquad (11)$$

Writing ξ and $\xi^\dagger$ explicitly from Eqs. (3) and (4), we see that

$$\left(1+\alpha^2\right)^{-1/2}$$

appears on both sides and may be cancelled, so Eq. (11) may be written

$$\begin{pmatrix}\alpha-i\alpha_3 & -\alpha_2-i\alpha_1\\ \alpha_2-i\alpha_1 & \alpha+i\alpha_3\end{pmatrix} M''$$

$$= \begin{pmatrix}\alpha+i\alpha_3 & \alpha_2+i\alpha_1\\ -\alpha_2+i\alpha_1 & \alpha-i\alpha_3\end{pmatrix} M' \quad . \qquad (12)$$

Let

$$M'' - M' = \begin{pmatrix}u_1 & u_2^*\\ u_2 & -u_1^*\end{pmatrix} \qquad (13)$$

and

$$M'' + M' = \begin{pmatrix}u_3 & u_4^*\\ u_4 & -u_3^*\end{pmatrix} \quad . \qquad (14)$$

Then from Eq. (12), we may write

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \frac{i}{\alpha}\begin{pmatrix}\alpha_3 & \alpha_1-i\alpha_2\\ \alpha_1+i\alpha_2 & -\alpha_3\end{pmatrix}\begin{pmatrix}u_3\\ u_4\end{pmatrix} \quad . \qquad (15)$$

To make the rotation angles complex, we let

$$\alpha \rightarrow i\alpha \quad .$$

It is shown in the appendix that Eq. (10) then becomes a Lorentz transformation and that the relative velocity of the frames divided by the velocity of light is

$$\beta = 2/(\alpha+1/\alpha) \quad . \qquad (16)$$

The transformation of Eq. (15) becomes

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}\alpha_3/\alpha & \left(\alpha_1-i\alpha_2\right)/\alpha\\ \left(\alpha_1+i\alpha_2\right)/\alpha & -\alpha_3/\alpha\end{pmatrix}\begin{pmatrix}u_3\\ u_4\end{pmatrix} \qquad (17)$$

and its inverse

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \alpha\alpha_3 & \alpha(\alpha_1 - i\alpha_2) \\ \alpha(\alpha_1 + i\alpha_2) & -\alpha\alpha_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . \qquad (18)$$

If the α_j are the direction cosines of the momentum, $\alpha > 1$, and

$$\beta = 2/(\alpha + 1/\alpha) \quad , \quad \text{then}$$

Eqs. (17) and (18) are just Eqs. (1) and (2), since

$$|p|\, c/\left(E - m_0\, c^2\right) = \beta/\left[1 - \left(1 - \beta^2\right)^{1/2}\right] =$$

$$\frac{2/(\alpha+1/\alpha)}{[1-(\alpha-1/\alpha)/(\alpha+1/\alpha)]} = \alpha \quad . \qquad (19)$$

Similarly, $|p|c/\left(E + m_0\, c^2\right) = 1/\alpha$. (20)

The physical significance of the imaginary rotation in real space is obscure. A real vector quantity in the frame corresponding to M is described as complex in M', after

$$\alpha \to i\alpha$$

is substituted in Eq. (9).

If we let

$$\alpha \to -i\alpha$$

in Eq. (16), it becomes

$$\beta = i2/(\alpha - 1/\alpha) \quad ,$$

i. e., Eq. (15) could be a wave equation for a tachyon-like[3] object, but one which has a real mass and imaginary momentum.

The quantity α is more informative than β. There are two values of α for each value of β, one less than one and one greater than one. A particle for which $\alpha < 1$ can never have a value of α greater than 1 because it must attain the velocity of light to pass through 1. For an electron, $\alpha < 1$, while a positron has $\alpha > 1$. The transformation $\alpha \to 1/\alpha$ is charge conjugation.

Potentials may be included in α, i.e., if

$$\alpha = \frac{|\hat{p}c - e\hat{A}|}{E - eV - m_0\, c^2} \quad , \qquad (21)$$

the expression

$$\beta = 2/(\alpha + 1/\alpha)$$

is still relativistically correct (Ref. 1, p. 58), and Eqs. (17) and (18) become the Dirac equation with potentials.

This geometric interpretation of the Dirac equation resembles the picture of space axes and body axes hypothesized by Corben,[4] who generated Lorentz invariant operators, possibly suitable for describing elementary particles, by projecting the angular momentum of a spinning top onto axes fixed in the body of the top.

REFERENCES

1. J. Eisele, Modern Quantum Mechanics with Applications to Elementary Particles, (Wiley-Interscience, New York, 1969), p. 261.
2. M. Naimark, Linear Representations of the Lorentz Group, translated by A. Swinfen and O. Marstrand, (MacMillan, New York, 1964), p. 13.
3. O.-M. Bilaniuk and E. C. G. Sudarshan, "Particles Beyond the Light Barrier," Physics Today 22, 43-51 (May 1969).
4. H. Corben, Classical and Quantum Theories of Spinning Particles (Holden-Day, San Francisco, 1968).

APPENDIX

Proof that the four-dimensional rotation defined by $M'' = \xi^\dagger M \xi^\dagger$ is a Lorentz transformation when the rotation angle is complex.

We may set $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$ in the expression for ξ_0 and ξ_1, so

$$\xi_0 = \alpha/\left(1+\alpha^2\right)^{1/2}$$
$$\xi_1 = -i/\left(1+\alpha^2\right)^{1/2} \qquad (A1)$$

Then $M'' = \xi^{\dagger} M \xi^{\dagger}$ may be written

$$\begin{pmatrix} x''_3 + ix''_4 & x''_1 - ix''_2 \\ x''_1 + ix''_2 & -x''_3 + ix''_4 \end{pmatrix}$$

$$= \left(\frac{1}{1+\alpha^2}\right)\begin{pmatrix} \alpha & i \\ i & \alpha \end{pmatrix}\begin{pmatrix} x_3+ix_4 & x_1-ix_2 \\ x_1+ix_2 & -x_3+ix_4 \end{pmatrix}\begin{pmatrix} \alpha & i \\ i & \alpha \end{pmatrix}. \qquad (A2)$$

Expanding, we have

$$\begin{bmatrix} x''_1 \\ x''_2 \\ x''_3 \\ x''_4 \end{bmatrix} = \begin{bmatrix} \frac{\alpha-1/\alpha}{\alpha+1/\alpha} & 0 & 0 & \frac{-2}{\alpha+1/\alpha} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{\alpha+1/\alpha} & 0 & 0 & \frac{\alpha-1/\alpha}{\alpha+1/\alpha} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \qquad (A3)$$

After the transformation $\alpha \to i\alpha$, Eq. (A3) becomes

$$\begin{bmatrix} x''_1 \\ x''_2 \\ x''_3 \\ x''_4 \end{bmatrix} = \begin{bmatrix} \frac{\alpha+1/\alpha}{\alpha-1/\alpha} & 0 & 0 & \frac{2i}{\alpha-1/\alpha} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-2i}{\alpha-1/\alpha} & 0 & 0 & \frac{\alpha+1/\alpha}{\alpha-1/\alpha} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \qquad (A4)$$

If $\beta = 2/(\alpha+1/\alpha)$ then $\gamma = 1/(1-\beta^2)^{1/2} = (\alpha+1/\alpha)/(\alpha-1/\alpha)$ and $\gamma\beta = 2/(\alpha-1/\alpha)$. Equation (A4) becomes

$$\begin{bmatrix} x''_1 \\ x''_2 \\ x''_3 \\ x''_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (A5)$$

This is the familiar form for a Lorentz transformation with coordinates x_1, x_2, x_3, $x_4 = ict$.