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HAMILTON'S PRINCIPLE AND NUMERICAL SOLUTION OF THE VLASOV EQUATIONS

(H. R. Lewis)

The Vlasov system of integro-partial differential equations, expressed in terms of Lagrangian variables for the particles and Eulerian variables for the electromagnetic field, can be derived from Hamilton's principle by using the Lagrangian that was given by Low[1] and by Sturrock.[2] That Lagrangian can be generalized to include systems in which there are present any of a large class of material media that exhibit nonlinear polarizability and magnetizability, and also to include forces of nonelectromagnetic character. These possibilities allow convenient imposition of certain kinds of particle and field boundary conditions for finite systems. Not only can the exact equations of motion for these systems be derived from Hamilton's principle, it can also be used to derive approximations to the exact equations of motion. The approximate system of equations to be emphasized here is a system of ordinary differential equations in time, although the basic method can be used to derive other approximate systems of equations--i.e., a system of difference equations. The method can be applied to any Lagrangian system.

The first step in solving equations approximately by any method is to choose the form in which the desired functions are to be represented. A common way of representing a function of continuous variables approximately is to specify the function on a mesh, i.e., to specify the function at each of a set of discrete values of the arguments. Another method of representation frequently used is to approximate the function by a linear combination of a finite number of linearly independent basis functions. The type of representation chosen is really a combination of these two, because the differential equations in time that are derived must generally be solved numerically, whereas the dependences of the unknown functions on other variables will be represented by linear combinations of linearly independent basis functions. Let $g(x,t)$ be one of the functions to be represented, where t is time and x denotes the remaining arguments of g. For each particular value of t, $g(x,t)$ may be approximated by a linear combination of a finite number of linearly independent basis functions, say $\psi_i(x)$. The coefficients in the expansion are functions of t only, so that the representation of $g(x,t)$ is

$$g(x,t) = \sum_{i=1}^{N} a_i(t)\psi_i(x). \qquad (1)$$

Suppose that all the unknown functions are approximated in this fashion; the problem then is to derive equations that determine the time evolution of the time-dependent coefficients and to solve those equations. The differential euqations that are chosen to determine the time evolution of the coefficients will clearly be ordinary differential equations, and, from the standpoint of numerical analysis, that in itself may be a distinct advantage over having to work with a system of integro-partial differential equations. By choosing the basis functions appropriately, the system of ordinary differential equations can be put into a form for which there are standard finite-difference methods of numerical solution whose properties are relatively well understood. At least one such method is formulated in a way that guarantees numerical stability.[3,4]

The mere fact of requiring that the unknown functions be approximated in the way illustrated by Eq. (1) does not specify the differential equations that determine the time evolution of the coefficients. In fact, there are infinitely many systems of equations for the coefficients that can be derived from the exact integro-differential equations. The solutions of these different systems of equations will differ in the fidelity with which they represent the time evolution of the unknown functions, and it is desirable to have a criterion for choosing one of the infinitely many systems. The method proposed for choosing a system is to substitute the approximate representations of the unknown functions into the Lagrangian density and then to determine the system of differential equations for the time-dependent coefficients by applying Hamilton's principle. The only difference between this procedure and using Hamilton's principle to derive the exact equations is that the functional variations are restricted to be within the class of functions chosen for approximating the unknown functions. In some useful sense, the system of equations so derived should approximate the exact equations as well as is possible.

It may be hoped that this procedure based on Hamilton's principle will prove particularly advantageous for nonlinear systems, such as the Vlasov system, and that some useful properties of the approximate system of equations can be derived. Details of the method are described in a LASL Report.[5]

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THE MOTION OF A TIME-DEPENDENT HARMONIC OSCILLATOR, AND THE MOTION OF A CHARGED PARTICLE IN A CLASS OF TIME-DEPENDENT, AXIALLY SYMMETRIC ELECTROMAGNETIC FIELDS

(H. R. Lewis)

The exact invariant described for a time-dependent harmonic oscillator[1,2] can be used to derive an elegant representation of the general solution of the equations of motion for the oscillator. This solution can be used to obtain the general solution of the equations of motion for a charged particle in certain electromagnetic fields.

The equations of motion for a time-dependent harmonic oscillator are equivalent to those for a particle moving in a certain type of electromagnetic field which is a superposition of two components. One component is a time-dependent, axially symmetric, uniform magnetic field and the associated induced electric field that corresponds to zero charge density. The other component is the radial electric field produced by a time-dependent, axially symmetric, uniform charge distribution. Because of the equivalence of the equations of motion, the representation of the general solution of the oscillator problem can be transcribed into a corresponding representation of the general solution of the particle problem.

Time-Dependent Harmonic Oscillator

A time-dependent harmonic oscillator is defined as a system described by the equation

$$\epsilon^2 \ddot{q} + \Omega^2(t) q = 0, \tag{1}$$

where time differentiation is denoted by a dot, and where $\Omega(t)$ is an arbitrary piecewise continuous function of time. It has been shown[1,2] that the function

$$I = \frac{1}{2}\left[\rho^{-2} q^2 + \epsilon^2 (\rho \dot{q} - q\dot{\rho})^2\right] \tag{2}$$

is an exact invariant of Eq. (1) as long as $\rho(t)$ is <u>any</u> particular solution of

$$\epsilon^2 \ddot{\rho} + \Omega^2(t)\rho - \rho^{-3} = 0. \tag{3}$$

The quantities q and Ω may be complex.

Equations (1) and (2) can be simplified significantly by replacing the variables q and t by variables Q and τ defined by

$$Q = \frac{q}{\rho}, \quad \tau = \frac{1}{\epsilon} \int^t \rho^{-2} (t')dt'. \tag{4}$$

It is easily verified that the expression for I in terms of Q and τ is

$$I = \frac{1}{2} [Q^2 + (\frac{dQ}{d\tau})^2]. \tag{2'}$$

The differential equation for Q as a function of τ corresponding to Eq. (1) is

$$\frac{d^2Q}{d\tau^2} + Q = 0. \tag{5}$$

The general solution of Eq. (5) is

$$Q = Ae^{i\tau} + Be^{-i\tau}, \tag{6}$$

where A and B are arbitrary complex constants, and I, A, and B are related by

$$I = 2AB.$$

Therefore, the general solution of Eq. (5) can be written as

$$Q = \sqrt{\frac{1}{2} I} \; [\sigma e^{i(\tau-\tau_o)} + \frac{1}{\sigma} e^{-i(\tau-\tau_o)}], \tag{7}$$

where $\sigma e^{i\tau_o} = \sqrt{A/B}$, and σ and τ_o are arbitrary real constants. Equation (7) is an elegant representation of the general solution of Eq. (1). There is one representation for each ρ that satisfies Eq. (3).

Charged Particle

Consider a particle of mass m and charge e moving in an electromagnetic field defined by the potentials

$$\left.\begin{aligned} \varphi &= \frac{1}{2} \frac{e}{mc^2} \eta(t) r^2 = \frac{1}{2} \frac{e}{mc^2} \eta(t) \; (x^2 + y^2) \\ &\text{and} \\ \underline{A} &= \frac{1}{2} B(t) \; \hat{k} \times \underline{r}, \end{aligned}\right\} \tag{8}$$

where $\underline{r}$ is the position vector, $\hat{k}$ is a unit vector along the symmetry axis, r is distance from the symmetry axis, x and y are cartesian coordinates perpendicular to the symmetry axis, and $\eta(t)$ and B(t) are arbitrary piecewise continuous functions. The electric and magnetic fields are

$$\underline{E} = -\nabla\varphi - \frac{1}{c}\dot{\underline{A}}$$

$$= -\frac{e}{mc^2} \eta(t) \; (x\hat{i} + y\hat{j}) - \frac{1}{2c} \dot{B}(t) \; \hat{k} \times \underline{r}$$

$$\underline{B} = \nabla \times \underline{A}$$

$$= B(t)\hat{k},$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the positive x and y directions, respectively. The equations of motion for the particle are

$$\left.\begin{aligned} \epsilon\ddot{x} &= -\frac{1}{\epsilon} \eta(t)x + \frac{1}{2} \dot{B}(t)y + B(t)\dot{y} \\ \epsilon\ddot{y} &= -\frac{1}{\epsilon} \eta(t)y - \frac{1}{2} \dot{B}(t)x - B(t)\dot{x}, \end{aligned}\right\} \tag{9}$$

where

$$\epsilon = \frac{mc}{e} .$$

The equations of motion can be written simply in terms of a complex variable, q, defined by

$$re^{i\theta} = x + iy = q \, e^{-\frac{i}{2\epsilon}\int^t B(t')dt'} . \tag{10}$$

The quantities r and θ are the usual cylindrical coordinates of the particle. The equation satisfied by q is

$$\left.\begin{aligned} &\epsilon^2 \ddot{q} + \Omega^2(t) q = 0, \\ &\text{where} \\ &\Omega^2(t) = \frac{1}{4} B^2(t) + \eta(t) \end{aligned}\right\} . \tag{11}$$

The function $\Omega^2(t)$ may be negative.

Since Eq. (11) is that of a time-dependent harmonic oscillator, we can now combine Eqs. (7) and (10) to obtain expressions for r and θ as functions of t. The result is

$$\left.\begin{aligned} &R = \frac{r}{\rho} = |Q| \\ &\text{and} \\ &\theta = \arg Q - \frac{1}{2\epsilon} \int^t B(t')dt', \end{aligned}\right\} \tag{12}$$

where ρ is <u>any</u> solution of Eq. (3), τ is defined by Eq. (4), and where

$$Q = \sqrt{\frac{1}{2}\, I}\left[\sigma e^{\,i(\tau-\tau_o)} + \frac{1}{\sigma} e^{\,-i(\tau-\tau_o)}\right]$$

$$|Q|^2 = \frac{1}{2}|I|\frac{(\sigma^2+1)^2}{\sigma^2}\left[\cos^2(\tau-\tau_o) + \left(\frac{\sigma^2-1}{\sigma^2+1}\right)^2 \sin^2(\tau-\tau_o)\right]$$

$$\arg Q = \tan^{-1}\left[\left(\frac{\sigma^2-1}{\sigma^2+1}\right)\tan(\tau-\tau_o)\right] - \frac{1}{2}\arg I$$

$$I = \frac{1}{2} e^{2i\left[\theta + \frac{1}{2\epsilon}\int^t B(t')dt'\right]}\left\{\left[R^2 + \left(\frac{dR}{d\tau}\right)^2 - \left(\epsilon\frac{p_\theta}{m}\right)^2\frac{1}{R^2}\right] + 2i\epsilon\frac{p_\theta}{m}\frac{1}{R}\frac{dR}{d\tau}\right\}$$

$$|2I|^2 = \left[R^2 + \left(\frac{dR}{d\tau}\right)^2 - \left(\epsilon\frac{p_\theta}{m}\right)^2\frac{1}{R^2}\right]^2 + 4\epsilon^2\left(\frac{p_\theta}{m}\right)^2\frac{1}{R^2}\left(\frac{dR}{d\tau}\right)^2 \qquad (13)$$

$$\arg I = 2\left[\theta + \frac{1}{2\epsilon}\int^t B(t')dt'\right] + \tan^{-1}\left\{\frac{2\epsilon\frac{p_\theta}{m}\frac{1}{R}\frac{dR}{d\tau}}{R^2 + \left(\frac{dR}{d\tau}\right)^2 - \left(\epsilon\frac{p_\theta}{m}\right)^2\frac{1}{R^2}}\right\}$$

$$\frac{1}{m}p_\theta = r^2\left[\dot\theta + \frac{1}{2\epsilon}B(t)\right]$$

$$\sigma = 2^{-\frac{1}{2}}\left\{\frac{2\epsilon}{m}\frac{p_\theta}{|I|} + \sqrt{\left(\frac{2\epsilon}{m}\frac{p_\theta}{|I|}\right)^2 + 4}\right\}^{\frac{1}{2}} .$$

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