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TWO-POINT CORRELATION EQUATIONS FOR VARIABLE DENSITY TURBULENCE

by

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ABSTRACT

A complete set of two-point correlation equations for variable-density turbulence is derived to consistent order in mass-weighted variables (Favre averaging). The derivation is based on a two-point generalization of the Reynolds stress tensor. The equations are transformed with respect to the separation between the two points to Fourier space. The correlation equations, as well as the Fourier-transformed equations, provide insights that are unavailable in the one-point equations. The derivation of spectral closures is significantly more complicated than that of constant-density closures or one-point variable-density closures due to the complex nature of isotropic scalar-vector correlation functions for nonsolenoidal fields. Several necessary constraints for the correlation functions are presented. In addition, a simple spectral model that satisfies these constraints is presented for illustrative purposes, and a discussion of the two-point correlations and their relationship to the corresponding correlations arising in one-point derivations is provided.

I. INTRODUCTION

The theoretical understanding of the turbulence of fluids with large-density fluctuations is in its infancy. In the case of constant-density turbulence, we have made much progress in understanding the nature of the large-scale structure (e.g., Saffman, 1967), the inertial range dynamics (e.g., Kolmogorov, 1941, 1962; Kraichnan, 1964a,b; Onsager, 1945; Heisenberg, 1948; Grant et al., 1962; Kida, 1987; Yakhot et al., 1989), energy transfer in Fourier space (e.g., Kraichnan, 1971, 1987; Lesieur and Schertzer, 1978; Domaradzki and Rogallo, 1990; Waleffe, 1992), and dissipation range dynamics (e.g., Yakhot et al., 1989; Kraichnan, 1964a), relatively little can be said regarding the variable-density case. The paucity of theoretical results and models of variable-density turbulence is due to the complexity introduced by the fluctuating density.

In spite of its difficulties, variable-density turbulence remains an important area of research, having relevance to many areas of industrial interest. As a consequence, a number of engineering models (one-point statistical models) have been proposed, e.g., the Reynolds stress-ε (Besnard et al., 1992; Cranfill, 1992). These models, primarily intended to describe multimaterial compressible flow, represent extensions of concepts from incompressible constant-density turbulence modeling and incorporate ideas from multiphase flow models.

In this report we seek to derive a consistent set of unclosed two-point correlation equations for turbulent flows with large density fluctuations, beginning from a particular generalization of the Reynolds stress tensor in mass-weighted variables. Some interesting properties of the resulting correlations will be highlighted. The resulting equations will be then recast for the case of homogeneous incompressible variable-density flow driven by a simple homogeneous pressure gradient. Finally, a simple closure of these equations will be presented for illustrative purposes.

II. A DESCRIPTION OF TURBULENCE WITH LARGE DENSITY FLUCTUATIONS

A. Governing Equations

For constant-density turbulence, the fluid flow may be assumed to be exactly described by the Navier-Stokes (NS) equations with appropriate boundary conditions. For engineering applications the complexity of the solutions of the NS equations usually precludes a numerical solution of them and, instead, the so-called Reynolds-averaged Navier-Stokes equations are solved, wherein the averaged velocity is computed, and the effect of the fluctuations (i.e., the turbulence) is manifested in the Reynolds stress tensor.

For the case of fluids with density fluctuations, additional equations are needed to describe the evolution of the density field, and an internal energy equation is needed for the case of compressible flow. The equations for momentum, $\rho \mathbf{u}$; density, ρ; and internal energy, I; are

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_n}{\partial x_n} = \frac{\partial \sigma_{in}}{\partial x_n} + \rho g_i, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_n}{\partial x_n} = 0, \tag{2.2}$$

$$\frac{\partial \rho I}{\partial t} + \frac{\partial \rho I u_n}{\partial x_n} = \sigma_{nm} \frac{\partial u_n}{\partial x_m} - \frac{\partial Q_n}{\partial x_n}, \tag{2.3}$$

where Q_n is the heat flux,

$$Q_n = -\kappa \frac{\partial T}{\partial x_n},$$

κ is the thermal conductivity, T is the temperature, and σ_{ij} is the total stress tensor:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}, \tag{2.4}$$

and

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_n}{\partial x_n}\right). \tag{2.5}$$

Fick's law for diffusion with a constant diffusion coefficient, D, is

$$\frac{\partial \rho c_\alpha}{\partial t} + \frac{\partial \rho u_n c_\alpha}{\partial x_n} = \frac{\partial}{\partial x_n}\left(\rho D \frac{\partial c_\alpha}{\partial x_n}\right), \tag{2.6}$$

where c_a is the mass fraction of species α. Note that the concentration equation leads to the following result for incompressible (low Mach number) flows:

$$\frac{\partial u_n}{\partial x_n} = -\frac{\partial}{\partial x_n}\left(\frac{D}{\rho}\frac{\partial \rho}{\partial x_n}\right). \tag{2.7}$$

For present purposes we assume that the fluid is incompressible, i.e., the density is not a direct function of the pressure. The internal energy equation may be neglected, and Eq. (2.7) provides an additional constraint. Note that the velocity field of this *incompressible* flow is *not solenoidal*. Equation (2.7) leads to a diffusive form of the density equation for incompressible mixing (as distinguished from the conservation form given in Eq. (2.2) for compressible and incompressible mixing):

$$\frac{\partial \rho}{\partial t} + u_n \frac{\partial \rho}{\partial x_n} = \rho \frac{\partial}{\partial x_n}\left(\frac{D}{\rho}\frac{\partial \rho}{\partial x_n}\right). \tag{2.8}$$

An evolution for the specific volume v, i.e., the inverse of the density,

$$v = \frac{1}{\rho}, \tag{2.9}$$

is useful. Note that because ρ is positive indefinite, then ν is also positive indefinite. From the conservation form of the density equation the ν-equation is

$$\frac{\partial \nu}{\partial t}+\frac{\partial \nu u_n}{\partial x_n}=2\nu\frac{\partial u_n}{\partial x_n}. \tag{2.10}$$

An incompressible diffusive form of the ν-equation is

$$\frac{\partial \nu}{\partial t}+u_n\frac{\partial \nu}{\partial x_n}=\nu\frac{\partial}{\partial x_n}\left(\frac{D}{\nu}\frac{\partial \nu}{\partial x_n}\right). \tag{2.11}$$

B. Mass-Weighted Averaging and the Fluctuation Equations

Following the usual conventions of turbulence modeling, we decompose the equations into averaged quantities and fluctuating quantities. The averaging procedure will be that introduced by Favre since this mass-weighted averaging method leads to a conservative form for the Reynolds stress tensor in the averaged momentum equation. The decompositions are

$$\rho=\overline{\rho}+\rho', \tag{2.12a}$$

$$\nu=\overline{\nu}+\nu', \tag{2.12b}$$

$$P=\overline{P}+P', \tag{2.12c}$$

$$u_i=\tilde{U}_i+u_i'', \tag{2.12d}$$

and

$$u_i=\overline{U}_i+u_i', \tag{2.12e}$$

where the overbar denotes a non-mass-weighted ensemble average, a single prime denotes a fluctuation about the non-mass-weighted average, the tilde denotes a mass-weighted ensemble average, and the double prime denotes a fluctuation about the mass-weighted average. The mass weighting of the velocity possesses the following traits:

$$\tilde{u}_i=\frac{\overline{\rho u_i}}{\overline{\rho}}, \tag{2.13a}$$

$$\overline{\rho u_i''}=0, \tag{2.13b}$$

and

$$\overline{\rho' u_i'} = \overline{\rho'\left(\tilde{U}_i + u_i'' - \overline{U}_i\right)} = \overline{\rho' u_i''}. \tag{2.13c}$$

The mass-weighted fluctuating velocity is related to the non-mass-weighted fluctuating velocity

$$u_i'' = a_i + u_i', \tag{2.13d}$$

where

$$a_i = \overline{u_i''}. \tag{2.13e}$$

A one-point density velocity correlation may be defined as[1]

$$\overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\overline{\rho}} = -\frac{\overline{\rho' u_i'}}{\overline{\rho}} = -\frac{\overline{\rho u_i'}}{\overline{\rho}} = +a_i \neq 0 \tag{2.13f}$$

When the NS equation is decomposed and averaged, we get

$$\frac{\partial \overline{\rho} \tilde{U}_i}{\partial t} + \frac{\partial \overline{\rho} \tilde{U}_i \tilde{U}_n}{\partial x_n} = \frac{\partial}{\partial x_n}\left(\overline{\sigma}_{in} - R_{in}\right) + \overline{\rho} g_i, \tag{2.14}$$

where

$$R_{ij} = \overline{\rho u_i'' u_j''}, \tag{2.15}$$

is the Reynolds stress tensor in the mass-weighted average variables. Two forms of the equation for the fluctuating velocities will be useful:

$$\frac{\partial u_i''}{\partial t} + u_n'' \frac{\partial}{\partial x_n}\left(\tilde{U}_i + u_i''\right) + \tilde{U}_n \frac{\partial u_i''}{\partial x_n} = \left(\overline{v} - \frac{1}{\overline{\rho}}\right)\frac{\partial \overline{\sigma}_{ni}}{\partial x_n} + v' \frac{\partial \overline{\sigma}_{ni}}{\partial x_n} + v \frac{\partial \sigma_{ni}'}{\partial x_n} + \frac{1}{\overline{\rho}}\frac{\partial R_{ni}}{\partial x_n}, \tag{2.16a}$$

and

$$\frac{\partial \rho u_i''}{\partial t} + \frac{\partial}{\partial x_n}\left[\rho u_i''\left(\tilde{U}_n + u_n''\right)\right] + \rho u_n'' \frac{\partial \tilde{U}_i}{\partial x_n} = \frac{\partial \sigma_{ni}'}{\partial x_n} - \frac{\rho'}{\overline{\rho}}\frac{\partial \overline{\sigma}_{ni}}{\partial x_n} + \left(1 + \frac{\rho'}{\overline{\rho}}\right)\frac{\partial R_{ni}}{\partial x_n}. \tag{2.16b}$$

When the conservation form of the density equation is decomposed and averaged, we get

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho} \tilde{U}_n}{\partial x_n} = 0, \tag{2.17}$$

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[1]Note that this definition of a uses the opposite sign than that used by Besnard et al., 1992.

and the corresponding conservation-form equation for the fluctuating density is

$$\frac{\partial \rho'}{\partial t}+\frac{\partial}{\partial x_n}\left(\rho'\tilde{U}_n+\rho u''\right)=0. \tag{2.18}$$

The decomposed and averaged specific-volume equation is

$$\frac{\partial \bar{v}}{\partial t}+\frac{\partial \tilde{U}_n\bar{v}}{\partial x_n}+\frac{\partial a_n\bar{v}}{\partial x_n}+\frac{\partial \overline{u_n''v'}}{\partial x_n}=2\left(\bar{v}\frac{\partial \tilde{U}_n}{\partial x_n}+\bar{v}\frac{\partial a_n}{\partial x_n}+\overline{v\frac{\partial u_n''}{\partial x_n}}\right), \tag{2.19}$$

and the fluctuating specific-volume equation is

$$\frac{\partial v'}{\partial t}+\frac{\partial v'\tilde{U}_n}{\partial x_n}=2v'\frac{\partial \tilde{U}_n}{\partial x_n}+2\left(v\frac{\partial u_n''}{\partial x_n}-\overline{v\frac{\partial u_n''}{\partial x_n}}\right)-\frac{\partial\left(vu_n''-\overline{vu_n''}\right)}{\partial x_n}. \tag{2.20}$$

A useful equation relating the averaged density and specific volume may be derived from Eq. (2.9):

$$\rho v=(\bar{\rho}+\rho')(\bar{v}+v')=1, \tag{2.21}$$

and averaging gives

$$\bar{\rho}\bar{v}-b=1, \tag{2.22}$$

where

$$b=-\overline{\rho' v'}. \tag{2.23}$$

C. The Two-Point Turbulent Stress Tensor

As shown in Eqs. (2.14) and (2.24), the turbulent stress tensor, R_{ij}, is the average of the product of three field variables—the density and two fluctuating velocities:

$$R_{ij}(\mathbf{x},t)=\overline{\rho(\mathbf{x},t)u_i''(\mathbf{x},t)u_j''(\mathbf{x},t)}. \tag{2.24}$$

In contrast, the Reynolds stress tensor, $\mathcal{R}_{ij}$, appearing in the Reynolds-averaged constant-density NS equations, which are the average of the product of two fluctuating velocities alone:

$$\mathcal{R}_{ij}(\mathbf{x},t)=\overline{u_i'(\mathbf{x},t)u_j'(\mathbf{x},t)}. \tag{2.25}$$

In the latter case the generalization to a two-point statistical quantity is obvious; each fluctuating velocity is taken at a different point:

$$\mathcal{R}_{ij}(\mathbf{x}_1,\mathbf{x}_2,t)=\overline{u_i'(\mathbf{x}_1,t)u_j'(\mathbf{x}_2,t)}, \tag{2.26}$$

and possesses the symmetry property

$$\mathcal{R}_{nn}(\mathbf{x}_1,\mathbf{x}_2,t)=\mathcal{R}_{nn}(\mathbf{x}_2,\mathbf{x}_1,t) \tag{2.27}$$

for any flow, thus ensuring that the Fourier spectrum of the energy is always real. For R_{ij}, the appropriate generalization to a two-point statistic is less clear. Two obvious choices are

$$R_{ij}(\mathbf{x}_1,\mathbf{x}_2,t)=\overline{\rho(\mathbf{x}_1,t)u_i''(\mathbf{x}_1,t)u_j''(\mathbf{x}_2,t)}, \tag{2.28}$$

or

$$R_{ij}(\mathbf{x}_1,\mathbf{x}_2,t)=\overline{\rho(\mathbf{x}_2,t)u_i''(\mathbf{x}_1,t)u_j''(\mathbf{x}_2,t)}. \tag{2.29}$$

Both forms reduce to Eq. (2.15) when $\mathbf{x}_1 = \mathbf{x}_2$ but neither form manifestly satisfies the symmetry property when $\mathbf{x}_1$ and $\mathbf{x}_2$ are interchanged. However, a simple linear combination of these two generalizations does satisfy the symmetry property, thus ensuring that the energy spectrum is always real:

$$R_{ij}(\mathbf{x}_1,\mathbf{x}_2,t)\equiv R_{ij}^{(+)}(\mathbf{x}_1,\mathbf{x}_2,t)=\frac{1}{2}\overline{\left[\rho(\mathbf{x}_1,t)+\rho(\mathbf{x}_2,t)\right]u_i''(\mathbf{x}_1,t)u_j''(\mathbf{x}_2,t)}. \tag{2.30}$$

The superscript (+) indicates that the densities at two points are summed and distinguishes the correlation from a subsequent correlation wherein the difference of the two densities is involved. For convenience the superscript (–) will generally be suppressed and only the superscript (–) will be explicitly retained in all cases.[2]

This definition of the generalized turbulent stress tensor for variable-density turbulence also reduces to Eq. (2.15) when $\mathbf{x}_1 = \mathbf{x}_2$. There are, of course, many ways of constructing the generalized two-point tensor to satisfy the symmetry constraint and reduce to Eq. (2.15) when $\mathbf{x}_1 = \mathbf{x}_2$. However, it is not clear that other definitions will be superior to Eq. (2.30) and thus (2.30) will be used as a starting point for the derivations.

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[2]The Fourier spectra association with (2.30) will not necessarily be nonnegative definite nor will any of the spectra associated with subsequently derived correlations.

D. R_{ij} -Transport Equation

From Eqs. (2.16a–b), we can derive an exact unclosed transport equation for R_{ij} $(\mathbf{x}_1,\mathbf{x}_2)$:

$$\frac{\partial R_{ij}(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\frac{\partial}{\partial x_{1n}}\left[R_{ij}(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_1)\right]+\frac{\partial}{\partial x_{2n}}\left[R_{ij}(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_2)\right]$$

$$+R_{in}(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_j(\mathbf{x}_2)}{\partial x_{2n}}+R_{nj}(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_i(\mathbf{x}_1)}{\partial x_{1n}}$$

$$-\frac{1}{2}R_{ij}(\mathbf{x}_1,\mathbf{x}_2)\left[\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}}+\frac{\partial \tilde{U}_n(\mathbf{x}_1)}{\partial x_{1n}}\right]-\frac{1}{2}\left[H_{ij}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)+H_{ji}^{R(+)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$-\frac{1}{2}R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\left[\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}}-\frac{\partial \tilde{U}_n(\mathbf{x}_1)}{\partial x_{1n}}\right]-\frac{1}{2}\left[H_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)-H_{ji}^{R(-)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$+\frac{\partial T_{ijn}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}}+\frac{\partial T_{jin}^{R(+)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)}{\partial x_{2n}}$$

$$=\frac{1}{2}\left\{\Psi_{ij}^{R}(\mathbf{x}_1,\mathbf{x}_2)+\Psi_{ji}^{R}(\mathbf{x}_2,\mathbf{x}_1)\right\}$$

$$+\frac{1}{2}\left\{\left[a_i(\mathbf{x}_1,\mathbf{x}_2)-\frac{m_i(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}(\mathbf{x}_2)}\right]\frac{\partial \bar{\sigma}_{nj}(\mathbf{x}_2)}{\partial x_{2n}}+\left[a_j(\mathbf{x}_2,\mathbf{x}_1)-\frac{m_j(\mathbf{x}_2,\mathbf{x}_1)}{\bar{\rho}(\mathbf{x}_1)}\right]\frac{\partial \bar{\sigma}_{in}(\mathbf{x}_1)}{\partial x_{1n}}\right\}. \quad (2.31)$$

The new correlations appearing in Eq. (2.31) are

$$R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{2}\overline{\left[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)\right]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)}, \quad (2.32)$$

$$\Psi_{ij}^{R}(\mathbf{x}_1,\mathbf{x}_2)=\overline{u_i''(\mathbf{x}_1)\left[1+\frac{\rho(\mathbf{x}_1)}{\rho(\mathbf{x}_2)}\right]\frac{\partial \sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}, \quad (2.33)$$

$$a_i(\mathbf{x}_1,\mathbf{x}_2)=\overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v(\mathbf{x}_2)}, \quad (2.34)$$

$$m_i(\mathbf{x}_1,\mathbf{x}_2)=\overline{u_i''(\mathbf{x}_1)\rho'(\mathbf{x}_2)}, \tag{2.35}$$

and

$$H_{ij}^{R^{(+)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\frac{1}{2}\overline{[\rho(\mathbf{x}_1)+\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.36}$$

$$H_{ij}^{R^{(-)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\frac{1}{2}\overline{[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.37}$$

$$T_{ijn}^{R^{(+)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\frac{1}{2}\overline{[\rho(\mathbf{x}_1)+\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)u_n''(\mathbf{x}')}. \tag{2.38}$$

The first line of Eq. (2.31) represents the time rate of change and advection of R_{ij}. The second line is production and destruction of R_{ij} due to coupling with the mean flow. The third line represents production and destruction of R_{ij} due to mean field dilatation and contraction and due to dilatation and contraction of the fluctuating field. The fourth line is production and destruction of R_{ij} due to inhomogeneity in the dilatation and contraction of the mean field and the fluctuating field. Note that if the instantaneous velocity field is solenoidal (i.e., D = 0 and low Mach number), then lines three and four are each equal to zero. Line five is turbulent self-diffusion. Line six is the correlation of the fluctuating velocity, the fluctuating density, and the fluctuating pressure and viscous stresses, analogous to the pressure-strain rate correlations of constant-density turbulence. Line seven represents coupling of the turbulence with the mean pressure and mean viscous stresses, and line eight represents the coupling of the turbulence with the turbulent stresses.

The $R_{ij}^{(-)}$-correlations are active only if the turbulence is inhomogeneous. Also note that in the one-point limit (i.e., $\mathbf{x}_1$ = $\mathbf{x}_2$)

$$a_i(\mathbf{x},\mathbf{x})=-\frac{1}{\bar{\rho}(\mathbf{x})}m_i(\mathbf{x},\mathbf{x}). \tag{2.39}$$

Of the subsequent correlations appearing in Eq. (2.31), only a_i, m_i, $R_{ij}^{(-)}$, and $\Psi_{ij}^{R^{(+)}}$ are of the same order or of lesser order as the turbulent stress tensor. Thus transport equations will be derived for a_i, m_i and $R_{ij}^{(-)}$. The $\Psi_{ij}^{R^{(+)}}$ and H-terms will be modeled.

E. $R_{ij}^{(-)}$-Transport Equation

The $R_{ij}^{(-)}$-transport equation is derived in the same manner as the R_{ij}-equation:

$$\frac{\partial R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\frac{\partial}{\partial x_{1n}}\left[R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_1)\right]+\frac{\partial}{\partial x_{2n}}\left[R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_2)\right]$$

$$+R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_j(\mathbf{x}_2)}{\partial x_{2n}}+R_{nj}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_i(\mathbf{x}_1)}{\partial x_{1n}}$$

$$-\frac{1}{2}R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\left[\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}}+\frac{\partial \tilde{U}_n(\mathbf{x}_1)}{\partial x_{1n}}\right]-\frac{1}{2}\left[H_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)+H_{ji}^{R(-)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$-\frac{1}{2}R_{ij}(\mathbf{x}_1,\mathbf{x}_2)\left[\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}}-\frac{\partial \tilde{U}_n(\mathbf{x}_1)}{\partial x_{1n}}\right]-\frac{1}{2}\left[H_{ij}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)-H_{ji}^{R(+)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$+\frac{\partial T_{ijn}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}}+\frac{\partial T_{jin}^{R(-)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)}{\partial x_{2n}}$$

$$=\frac{1}{2}\left[\Psi_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2)-\Psi_{ji}^{R(-)}(\mathbf{x}_2,\mathbf{x}_1)\right]$$

$$+\frac{1}{2}\left\{\left[a_i(\mathbf{x}_1,\mathbf{x}_2)+\frac{m_i(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}(\mathbf{x}_2)}\right]\frac{\partial \bar{\sigma}_{nj}(\mathbf{x}_2)}{\partial x_{2n}}-\left[a_j(\mathbf{x}_2,\mathbf{x}_1)+\frac{m_j(\mathbf{x}_2,\mathbf{x}_1)}{\bar{\rho}(\mathbf{x}_1)}\right]\frac{\partial \bar{\sigma}_{in}(\mathbf{x}_1)}{\partial x_{1n}}\right\}$$

$$-\frac{1}{2}\left\{\left[a_i(\mathbf{x}_1,\mathbf{x}_2)+\frac{m_i(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}(\mathbf{x}_2)}\right]\frac{\partial R_{nj}(\mathbf{x}_2)}{\partial x_{2n}}-\left[a_j(\mathbf{x}_2,\mathbf{x}_{(})+\frac{m_j(\mathbf{x}_2,\mathbf{x}_1)}{\bar{\rho}(\mathbf{x}_1)}\right]\frac{\partial R_{in}(\mathbf{x}_1)}{\partial x_{1n}}\right\}. \quad (2.40)$$

The new correlations that arise are

$$\Psi_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2)=\overline{u_i''(\mathbf{x}_1)\left[\frac{\rho(\mathbf{x}_1)}{\rho(\mathbf{x}_2)}-1\right]\frac{\partial \sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}, \quad (2.41)$$

and

$$T_{inj}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\frac{1}{2}\overline{[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_n''(\mathbf{x}_2)u_j''(\mathbf{x}')}. \quad (2.42)$$

$T_{inj}^{R(-)}$ is of a higher order than the R_{ij} and thus no transport equation will be derived for it. The $\Psi_{ij}^{R(-)}$ couples the turbulence to the fluctuating pressure field and will be modeled. The terms of Eq. (2.40) are directly analogous to the corresponding terms in the R_{ij}-equation (2.31) and thus the same interpretations apply.

F. a_i-Transport Equation

An exact unclosed transport equation for a_i may be derived from Eqs. (2.16b), (2.19), and (2.20), noting that

$$a_i(\mathbf{x}_1,\mathbf{x}_2) = \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v(\mathbf{x}_2)} = \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)}. \tag{2.43}$$

The a_i-equation is

$$\begin{aligned}
&\frac{\partial a_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial t} + \frac{\partial}{\partial x_{1n}}\left[\tilde{U}_n(\mathbf{x}_1)a_i(\mathbf{x}_1,\mathbf{x}_2)\right] + \frac{\partial}{\partial x_{2n}}\left[\tilde{U}_n(\mathbf{x}_2)a_i(\mathbf{x}_1,\mathbf{x}_2)\right] \\
&+ a_n(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_i(\mathbf{x}_1)}{\partial x_{1n}} - 2\left[a_i(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}} + H_i^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)\right] \\
&- 2\left[R_{in}(\mathbf{x}_1,\mathbf{x}_2) + R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\right]\frac{\partial \overline{v}(\mathbf{x}_2)}{\partial x_{2n}} - \frac{\partial}{\partial x_{2n}}\left\{\overline{v}(\mathbf{x}_2)\left[R_{in}(\mathbf{x}_1,\mathbf{x}_2) + R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\right]\right\} \\
&+ \frac{\partial T_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)}{\partial x_{1n}} + \frac{\partial T_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}} \\
&= \frac{b(\mathbf{x}_1,\mathbf{x}_2)}{\overline{\rho}(\mathbf{x}_1)}\left[\frac{\partial \overline{\sigma}_{ni}(\mathbf{x}_1)}{\partial x_{1n}} - \frac{\partial R_{ni}(\mathbf{x}_1)}{\partial x_{1n}}\right] + \Psi_i^a(\mathbf{x}_2,\mathbf{x}_1).
\end{aligned} \tag{2.44}$$

The new correlations arising from the a_i-equation are

$$b(\mathbf{x}_1,\mathbf{x}_2) = -\overline{\rho'(\mathbf{x}_1)v'(\mathbf{x}_2)}, \tag{2.45}$$

$$\Psi_i^a(\mathbf{x}_1,\mathbf{x}_2) = \overline{v'(\mathbf{x}_1)\frac{\partial \sigma_{ni}'(\mathbf{x}_2)}{\partial x_{1n}}}, \tag{2.46}$$

$$H_i^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.47}$$

and

$$T_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{u_i'(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)u_n''(\mathbf{x}')}. \tag{2.48}$$

The first line of Eq. (2.44) represents the time rate of change and advection of a_i. The first term on the left side of the second line represents production and destruction due to coupling to mean flow gradients, the second term represents coupling to mean field and fluctuating field dilatation. Note that the terms in the [ ]-brackets in the second line will equal zero if the velocity field is solenoidal. The third line represents generation of a_i due to coupling of the turbulence with the density and specific volume gradients. The fourth line may represent a self-diffusion-type process. The first term on the left of the fifth line represents coupling of the turbulent field to the mean field pressure and viscous and turbulent stresses. The second term of the fifth line represents coupling of the fluctuating velocity and density fields with the fluctuating pressure and viscous stresses.

Note that of the subsequent turbulent correlations arising out of Eq. (2.44), only b and Ψ_i^a are of the same order or lesser order than the turbulent stress, R_{ij}. The correlations T_{in}^a and H_i^a are of the same order or higher order than the turbulent stress, e.g.,

$$T_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \bar{v}(\mathbf{x}_2)\left[R_{ij}(\mathbf{x}_1,\mathbf{x}') + R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}')\right] + \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)u_n''(\mathbf{x}')}, \tag{2.49}$$

where the first terms on the right-hand side are of the same order as R_{ij} and the second terms are of higher order. A transport equation will thus be derived for b. The terms Ψ_i^a, T_{in}^a, and H_i^a will be modeled.

G. m_i-Transport Equation

An exact unclosed transport equation for m_i may be derived from Eqs. (2.16a) and (2.18):

$$\frac{\partial m_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\frac{\partial}{\partial x_{1n}}\left[m_i(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_1)\right]+\frac{\partial}{\partial x_{2n}}\left[m_i(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_2)\right]$$

$$+\,m_n(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_i(\mathbf{x}_1)}{\partial x_{1n}}-m_i(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_n(\mathbf{x}_1)}{\partial x_{1n}}-H_i^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)$$

$$+\,\frac{\partial R_{in}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}-\frac{\partial R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}+\frac{\partial T_{in}^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)}{\partial x_{1n}}$$

$$=\;-b(\mathbf{x}_2,\mathbf{x}_1)\frac{\partial \bar{\sigma}_{ni}(\mathbf{x}_1)}{\partial x_{1n}}+\Psi_i^m(\mathbf{x}_2,\mathbf{x}_1), \tag{2.50}$$

where the additional correlations are

$$\Psi_i^m(\mathbf{x}_1,\mathbf{x}_2)=\overline{\rho'(\mathbf{x}_1)v(\mathbf{x}_2)\frac{\partial \sigma'_{ni}(\mathbf{x}_2)}{\partial x_{2n}}}, \tag{2.51}$$

$$H_i^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\rho'(\mathbf{x}_2)u_i''(\mathbf{x}_1)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.52}$$

and

$$T_{in}^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\rho'(\mathbf{x}_2)u_i''(\mathbf{x}_1)u_n''(\mathbf{x}')}. \tag{2.53}$$

The b-correlation in Eq. (2.50) is identical to that appearing in the a_i-equation and is defined in Eq. (2.45). The first line of Eq. (2.50) represents time rate of change of m_i and conservative advection (e.g., pure advection plus coupling to mean flow velocity gradients). The first term of the second line represents production and destruction of m_i due to coupling to mean-flow velocity gradients. The remaining two terms represent production and destruction of m_i due to mean-field dilatation and contraction and fluctuating field dilatation and contraction, respectively. The last two terms in the second line sum to zero if the instantaneous field is solenoidal. The third line describes turbulent self-diffusion and mixing, and the fourth line shows coupling of the turbulence with the pressure and viscous stresses. Note that H_i^m and T_{in}^m are of higher order than is R_{ij} and thus no transport equation is derived for them. The term Ψ_i^m involves the fluctuating pressure and fluctuating viscous effects and is modeled as is the H-term.

H. b - Transport Equation

The exact, unclosed b-transport equation is derived from Eqs. (2.18) and (2.20):

$$\frac{\partial b(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\frac{\partial}{\partial x_{1n}}\left[b(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_1)\right]+\frac{\partial}{\partial x_{2n}}\left[b(\mathbf{x}_1,\mathbf{x}_2)\tilde{U}_n(\mathbf{x}_2)\right]$$

$$-\,2\left[b(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial \tilde{U}_n(\mathbf{x}_2)}{\partial x_{2n}}-H^b(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)\right]$$

$$+\,\bar{v}(\mathbf{x}_2)\frac{\partial m_n(\mathbf{x}_2,\mathbf{x}_1)}{\partial x_{2n}}-m_n(\mathbf{x}_2,\mathbf{x}_1)\frac{\partial \bar{v}(\mathbf{x}_2)}{\partial x_{2n}}$$

$$-\,\frac{\partial a_n(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}}-\frac{\partial T_n^b(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{2n}}=0 \tag{2.54}$$

The new correlations are

$$H^b(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\rho'(\mathbf{x}_1)v'(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.55}$$

and

$$T_i^b(x_1,x_2;x')=\overline{\rho'(x_1)v'(x_2)u_i''(x')}. \tag{2.56}$$

These correlations are of higher order than R_{ij} and thus no transport equation will be derived for them. Note that the average specific volume (i.e., the inverse density) appears in the b-equation. Rather than carry an evolution equation for the average specific volume, we may use the simple algebraic relationship (2.22), yielding

$$\bar{v}(\mathbf{x})=\frac{1+b(\mathbf{x})}{\bar{\rho}(\mathbf{x})}. \tag{2.57}$$

The physical significance of the various terms arising in the b-equation is not immediately obvious, but the following interpretation of terms is defensible. The first line of Eq. (2.54) represents the time rate of change and the advection and production of b. The second line describes the production and destruction of turbulence due to mean-field dilatation and contraction (the first term) and fluctuating field dilatation and contraction (H^p). The first term on the left of the third line describes the production and destruction of b due to coupling of the mean-field inverse density to gradients of the turbulent field, and the second term describes the turbulent advection of the inverse density that causes production and destruction of b. The fourth line represents turbulent diffusion and advection of b.

I. Discussion

Exact, unclosed, transport equations have been derived for the two-point correlations of order equal to or less than the order of the generalized two-point turbulent stress tensor. Recall that the transported quantities are

$$R_{ij}(\mathbf{x}_1,\mathbf{x}_2,t) \equiv R_{ij}^{(+)}(\mathbf{x}_1,\mathbf{x}_2,t) = \frac{1}{2}\overline{[\rho(\mathbf{x}_1,t)+\rho(\mathbf{x}_2,t)]u_i''(\mathbf{x}_1,t)u_j''(\mathbf{x}_2,t)}, \tag{2.30}$$

$$R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{2}\overline{[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)}, \tag{2.32}$$

$$a_i(\mathbf{x}_1,\mathbf{x}_2) = \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v(\mathbf{x}_2)}, \tag{2.34}$$

$$m_i(\mathbf{x}_1,\mathbf{x}_2) = \overline{u_i''(\mathbf{x}_1)\rho'(\mathbf{x}_2)}, \tag{2.35}$$

and

$$b(\mathbf{x}_1,\mathbf{x}_2) = -\overline{\rho'(\mathbf{x}_1)v'(\mathbf{x}_2)}, \tag{2.45}$$

In the one-point limit, i.e., $\mathbf{x}_1 = \mathbf{x}_2$, these correlations reduce to

$$R_{ij}(\mathbf{x},\mathbf{x}) = \overline{\rho(\mathbf{x})u_i''(\mathbf{x})u_j''(\mathbf{x})}, \tag{2.58}$$

$$R_{ij}^{(-)}(\mathbf{x},\mathbf{x}) = 0, \tag{2.59}$$

$$a_i(\mathbf{x},\mathbf{x}) = \overline{u_i''(\mathbf{x})}, \tag{2.60}$$

$$m_i(\mathbf{x},\mathbf{x}) = -\bar{\rho}(\mathbf{x})\overline{u_i''(\mathbf{x})} = -\bar{\rho}(\mathbf{x})a_i(\mathbf{x},\mathbf{x}), \tag{2.61}$$

and

$$b(\mathbf{x},\mathbf{x}) = -\overline{\rho'(\mathbf{x})v'(\mathbf{x})}. \tag{2.62}$$

Thus, in the one-point limit, R_{ij} reduces to the usual turbulent stress tensor in Favre-averaged variables, $R_{ij}^{(-)}$ vanishes, and m_i is directly related to a_i. The one-point b and a_i

are identical to the turbulence quantities used by Besnard et al. (1992), except for the sign of a_i.

The additional correlations for which transport equations are not written can be placed into three categories: turbulent dilatation correlations (H-terms), triple correlations (T-terms), and fluctuating pressure-viscous strain correlations (Ψ-terms). Recall that the turbulent dilatation correlations are

$$H_{ij}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\frac{1}{2}[\rho(\mathbf{x}_1)+\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.36}$$

$$H_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\frac{1}{2}[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.37}$$

$$H_i^{a}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.47}$$

$$H_i^{m}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\rho'(\mathbf{x}_2)u_i''(\mathbf{x}_1)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}, \tag{2.52}$$

and

$$H^{b}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\rho'(\mathbf{x}_1)v'(\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}. \tag{2.55}$$

For a transport equation for a turbulence quantity, say $\bar{\phi}$, the form of the H-correlation is

$$H^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\phi(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial u_n''(\mathbf{x}')}{\partial x_n'}}. \tag{2.63}$$

A consistent modeling is sought for this class of terms. Note that the divergence of the fluctuating velocity field is expressed in Favre-averaged variables that may be nonsolenoidal even if the instantaneous field is solenoidal. In addition, because this fluctuating field has a nonzero mean, it may be advantageous to decompose u_i'' further into a_i and u_i' and model the part involving the divergence of the zero-mean-velocity fluctuation. Thus the H-terms will be written as

$$H^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')=\overline{\phi(\mathbf{x}_1,\mathbf{x}_2)}\frac{\partial a_n(\mathbf{x}')}{\partial x_n'}+\hat{H}^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}'), \tag{2.64}$$

where

$$\hat{H}^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{\phi(\mathbf{x}_1,\mathbf{x}_2)\frac{\partial u_n'(\mathbf{x}')}{\partial x_n'}}. \tag{2.65}$$

If the instantaneous field is solenoidal (e.g., mixing of immiscible fluids), then the $\hat{H}^{\bar{\phi}}$-terms are zero.

Recall that the triple correlations are

$$T_{ijn}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \frac{1}{2}\overline{[\rho(\mathbf{x}_1)+\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_j''(\mathbf{x}_2)u_n''(\mathbf{x}')}, \tag{2.38}$$

$$T_{inj}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \frac{1}{2}\overline{[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]u_i''(\mathbf{x}_1)u_n''(\mathbf{x}_2)u_j''(\mathbf{x}')}, \tag{2.42}$$

$$T_{in}^{a}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{u_i''(\mathbf{x}_1)\rho(\mathbf{x}_1)v'(\mathbf{x}_2)u_n''(\mathbf{x}')}, \tag{2.48}$$

$$T_{in}^{m}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{\rho'(\mathbf{x}_2)u_i''(\mathbf{x}_1)u_n''(\mathbf{x}')}, \tag{2.53}$$

and

$$T_{i}^{b}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{\rho'(\mathbf{x}_1)v'(\mathbf{x}_2)u_i''(\mathbf{x}')}. \tag{2.56}$$

The form of the triple correlations is

$$T_{\dots n}^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{\phi_{\dots}(\mathbf{x}_1,\mathbf{x}_2)u_n''(\mathbf{x}')} = \overline{\phi_{\dots}(\mathbf{x}_1,\mathbf{x}_2)}\, a_n(\mathbf{x}') + \hat{T}_{\dots n}^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}'), \tag{2.66}$$

where

$$\hat{T}_{\dots n}^{\bar{\phi}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}') = \overline{\phi_{\dots}(\mathbf{x}_1,\mathbf{x}_2)u_n'(\mathbf{x}')}. \tag{2.67}$$

The fluctuating pressure–viscous stress correlations, Ψ, may be rewritten in a "standardized" form. From Eq. (2.33),

$$\Psi_{ij}^{R}(\mathbf{x}_1,\mathbf{x}_2) = \overline{u_i''(\mathbf{x}_1)[\rho(\mathbf{x}_1)+\rho(\mathbf{x}_2)]v(\mathbf{x}_2)\frac{\partial\sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}, \tag{2.68}$$

from Eq. (2.41),

$$\Psi_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2) = \overline{u_i''(\mathbf{x}_1)[\rho(\mathbf{x}_1)-\rho(\mathbf{x}_2)]\nu(\mathbf{x}_2)\frac{\partial\sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}, \tag{2.69}$$

from Eq. (2.46)

$$\Psi_j^{a}(\mathbf{x}_1,\mathbf{x}_2) = \overline{\rho(\mathbf{x}_2)\nu(\mathbf{x}_1)\nu(\mathbf{x}_2)\frac{\partial\sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}, \tag{2.70}$$

and from Eq. (2.53),

$$\Psi_j^{m}(\mathbf{x}_1,\mathbf{x}_2) = \overline{\rho'(\mathbf{x}_1)\nu(\mathbf{x}_2)\frac{\partial\sigma_{nj}'(\mathbf{x}_2)}{\partial x_{2n}}}. \tag{2.71}$$

Unlike their constant density analog, the Ψ-correlations may be decomposed into a product of two nonzero terms,

$$\begin{aligned}\Psi_{i,\dots,j}^{\bar{\varphi}}(\mathbf{x}_1,\mathbf{x}_2) &= \overline{[\overline{\varphi}_{i\dots}(\mathbf{x}_1,\mathbf{x}_2)+\varphi_{i\dots}'(\mathbf{x}_1,\mathbf{x}_2)][\overline{\pi}_j(\mathbf{x}_2)+\pi_j'(\mathbf{x}_2)]} \\ &= \overline{\varphi}_{i\dots}(\mathbf{x}_1,\mathbf{x}_2)\overline{\pi}_j(\mathbf{x}_2)+\overline{\varphi_{i\dots}'(\mathbf{x}_1,\mathbf{x}_2)\pi_j'(\mathbf{x}_2)},\end{aligned} \tag{2.72}$$

where

$$\pi_j(\mathbf{x}) = \overline{\nu(\mathbf{x})\frac{\partial\sigma_{nj}'(\mathbf{x})}{\partial x_n}}, \tag{2.73}$$

and $\varphi_{i\dots}(\mathbf{x}_1,\mathbf{x}_2)$ is expressible in terms of the two-point variables already defined. In the limit as density variations approach zero, π_i and $\varphi_{i\dots}(\mathbf{x}_1,\mathbf{x}_2)$ become zero mean, the $\Psi^{R(+)}$ correlation becomes identical to its constant-density analog, and the additional Ψ-correlations vanish. These data suggest that the product of the means represents a phenomenon unique to the case of variable-density turbulence, and the correlation of the fluctuating parts represents analogs to the constant-density case. The constant-density case typically is modeled as a "slow part" governed by triple correlations and causing a tendency toward isotropy, and a "rapid part" governing the coupling of the turbulence to mean-flow gradients.

III. ISOTROPIC TURBULENCE AND HOMOGENEOUS ACCELERATED TURBULENCE

A. Isotropy and Homogeneity

The theoretical study of the turbulence-transport equations is made more tractable when simplifying assumptions are made regarding the geometry of the turbulence domain and of the turbulence. The assumptions usually applied in theoretical studies of constant turbulence are of statistical isotropy or statistical homogeneity. Statistically isotropic turbulence is defined as turbulence that is statistically invariant under translation and rotation of the coordinate axes. Statistically homogeneous turbulence is invariant under coordinate translations but not necessarily invariant under coordinate orientation. In other words, an isotropic turbulence exhibits no statistically preferred orientation or position; homogeneous turbulence may possess a statistically preferred direction but not a statistically preferred position.

For present purposes we will examine variable-density turbulence under the assumption of statistical isotropy and a specific case of statistical homogeneity. The assumptions of statistical isotropy and homogeneity are the same for variable-density turbulence as they are for constant-density turbulence, i.e., *statistical invariance* under coordinate rotation and translation for isotropy, and statistical invariance under translation for homogeneity. The specific case of a homogeneous turbulence subjected to an acceleration will be examined. Because this case may be counterintuitive to some, it deserves further explanation. A physical model can be described as a tank with sides of length, L, in which fluids of varying densities are vigorously mixed to such a degree that at points away from the walls the turbulence is essentially isotropic. This tank is then accelerated (gravity is "turned on"). In this situation the denser fluids will flow toward the bottom of the tank and the lighter fluid moves toward the top of the tank. If the fluid is incompressible, there is no net volumetric flux of fluid from the bottom of the tank to the top or vice-versa and

$$\overline{U_i} = \tilde{U}_i + a_i = 0. \tag{3.1}$$

Note that the heavy fluid will eventually accumulate at the bottom of the tank as the fluids separate, thus producing gradients in the mean density and violating the assumption of homogeneity. However, for present purposes, we will assume that the tank is large (the limit as L approaches infinity) and the time for separation to affect the region of the tank of interest is large compared with any time scales of interest.

With the above assumptions in mind, we may simplify the mean-flow equations and the two-point turbulence correlation equations. Spatial arguments of single-point statistical quantities are omitted because these quantities are invariant under translation.

B. The Two-Point Correlation Equations for Homogeneous Accelerated Turbulence

For homogeneous accelerated turbulence, the single-point statistics and quantities are invariant under coordinate translation and thus all spatial gradients of these quantities are zero. The mean-velocity equation is

$$\frac{\partial \tilde{U}_i}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} + g_i, \tag{3.2}$$

the mean density is constant,

$$\frac{\partial \bar{\rho}}{\partial t} = 0, \tag{3.3}$$

and the mean-specific volume is given by

$$\frac{\partial \bar{v}}{\partial t} = 2\overline{v\frac{\partial u_n'}{\partial x_n}} = 2\hat{H}^{\bar{V}} = \frac{1}{\bar{\rho}}\frac{\partial b(x,x)}{\partial t}. \tag{3.4}$$

The Reynolds stress spectral tensor equation for this case becomes

$$\frac{\partial R_{ij}(\mathbf{x}_1,\mathbf{x}_2)}{\partial t} + \tilde{U}_n\left[\frac{\partial R_{ij}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial R_{ij}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}\right] \tag{3.5}$$

$$-\frac{1}{2}\left[\hat{H}_{ij}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2) + \hat{H}_{ji}^{R(+)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right] - \frac{1}{2}\left[\hat{H}_{ij}^{R(-)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2) - \hat{H}_{ji}^{R(-)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$+ a_n\frac{\partial R_{ij}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + a_n\frac{\partial R_{ji}(\mathbf{x}_2,\mathbf{x}_1)}{\partial x_{2n}} + \frac{\partial \hat{T}_{ijn}^{R(+)}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial \hat{T}_{jin}^{R(+)}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)}{\partial x_{2n}}$$

$$= \frac{1}{2}\left[\Psi_{ij}^R(\mathbf{x}_1,\mathbf{x}_2) + \Psi_{ji}^R(\mathbf{x}_2,\mathbf{x}_1)\right]$$

$$-\frac{1}{2}\left\{\left[a_i(\mathbf{x}_1,\mathbf{x}_2) - \frac{m_i(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}}\right]\frac{\partial \bar{P}}{\partial x_j} + \left[a_j(\mathbf{x}_2,\mathbf{x}_1) - \frac{m_j(\mathbf{x}_2,\mathbf{x}_1)}{\bar{\rho}}\right]\frac{\partial \bar{P}}{\partial x_i}\right\},$$

the $R_{ij}^{(-)}$-equation is

$$\frac{\partial R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial t} + \tilde{U}_n\left[\frac{\partial R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial R_{ij}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}\right] \tag{3.6}$$

$$- \frac{1}{2}\left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2) + \hat{H}_{ji}^{R^{(-)}}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right] - \frac{1}{2}\left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2) - \hat{H}_{ji}^{R^{(+)}}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)\right]$$

$$+ a_j\frac{\partial R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + a_i\frac{\partial R_{jn}^{(-)}(\mathbf{x}_2,\mathbf{x}_1)}{\partial x_{2n}} + \frac{\partial \hat{T}_{ijn}^{R^{(-)}}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial \hat{T}_{jin}^{R^{(-)}}(\mathbf{x}_2,\mathbf{x}_1;\mathbf{x}_1)}{\partial x_{2n}}$$

$$= \frac{1}{2}\left[\Psi_{ij}^{R^{(-)}}(\mathbf{x}_1,\mathbf{x}_2) - \Psi_{ji}^{R^{(-)}}(\mathbf{x}_2,\mathbf{x}_1)\right]$$

$$- \frac{1}{2}\left\{\left[a_i(\mathbf{x}_1,\mathbf{x}_2) + \frac{m_i(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}}\right]\frac{\partial \bar{P}}{\partial x_j} - \left[a_j(\mathbf{x}_2,\mathbf{x}_1) + \frac{m_j(\mathbf{x}_2,\mathbf{x}_1)}{\bar{\rho}}\right]\frac{\partial \bar{P}}{\partial x_i}\right\},$$

the a_i-equation is,

$$\frac{\partial a_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial t} + \tilde{U}_n\left[\frac{\partial a_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial a_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}\right] \tag{3.7}$$

$$- 2\hat{H}_i^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2) - \bar{v}\frac{\partial}{\partial x_{2n}}\left[R_{in}(\mathbf{x}_1,\mathbf{x}_2) + R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)\right]$$

$$+ 2a_n\frac{\partial a_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}} + \frac{\partial \hat{T}_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)}{\partial x_{1n}} + \frac{\partial \hat{T}_{in}^a(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{1n}}$$

$$= -\frac{b(\mathbf{x}_1,\mathbf{x}_2)}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} + \Psi_i^a(\mathbf{x}_2,\mathbf{x}_1),$$

the m_i-equation is

$$\frac{\partial m_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\tilde{U}_n\left[\frac{\partial m_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}}+\frac{\partial m_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}\right] \tag{3.8}$$

$$-\hat{H}_i^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)+\frac{\partial R_{in}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}-\frac{\partial R_{in}^{(-)}(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}+a_n\frac{\partial m_i(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}}+\frac{\partial \hat{T}_{in}^m(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_1)}{\partial x_{1n}}$$

$$= b(\mathbf{x}_2,\mathbf{x}_1)\frac{\partial \overline{P}}{\partial x_i}+\Psi_i^m(\mathbf{x}_2,\mathbf{x}_1),$$

and the b-equation is

$$\frac{\partial b(\mathbf{x}_1,\mathbf{x}_2)}{\partial t}+\tilde{U}_n\left[\frac{\partial b(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}}+\frac{\partial b(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}\right]. \tag{3.9}$$

$$+\,2\hat{H}^b(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)+\overline{v}\frac{\partial m_n(\mathbf{x}_2,\mathbf{x}_1)}{\partial x_{2n}}-\frac{\partial a_n(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{1n}}-a_n\frac{\partial b(\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2n}}-\frac{\partial \hat{T}_n^b(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)}{\partial x_{2n}}=0.$$

Equations (3.1)-(3.9) represent the complete set of unclosed equations for the homogeneous accelerated turbulence.

IV. TRANSFORMATION TO THE CENTER COORDINATE + RELATIVE COORDINATE FRAME

A. Coordinate Definition

The correlation equations will next be rewritten in terms of a center coordinate, **X**, and a relative coordinate, **r**:

$$X = \frac{1}{2}[\mathbf{x}_1 + \mathbf{x}_2], \tag{4.1a}$$

$$r = \mathbf{x}_1 - \mathbf{x}_2, \tag{4.1b}$$

so

$$\mathbf{x}_1 = \mathbf{X} + \frac{1}{2}\mathbf{r}, \tag{4.2a}$$

$$\mathbf{x}_2 = \mathbf{X} - \frac{1}{2}\mathbf{r}. \tag{4.2b}$$

The derivative operators become

$$\frac{\partial(\ )}{\partial x_{1i}} = \frac{\partial(\ )}{\partial X_n}\frac{\partial X_n}{\partial x_{1i}} + \frac{\partial(\ )}{\partial r_n}\frac{\partial r_n}{\partial x_{1i}} = \frac{1}{2}\frac{\partial(\ )}{\partial X_i} + \frac{\partial(\ )}{\partial r_i}, \tag{4.3a}$$

and

$$\frac{\partial(\ )}{\partial x_{2i}} = \frac{\partial(\ )}{\partial X_n}\frac{\partial X_n}{\partial x_{2i}} + \frac{\partial(\ )}{\partial r_n}\frac{\partial r_n}{\partial x_{2i}} = \frac{1}{2}\frac{\partial(\ )}{\partial X_i} - \frac{\partial(\ )}{\partial r_i}. \tag{4.3b}$$

The following notational convention will be adopted. A variable that is a function of two of the independent variables $\mathbf{x}_1$ or $\mathbf{x}_2$ becomes a function of **X** and of **r** if the arguments are in the order $(\mathbf{x}_1,\mathbf{x}_2)$ or $-\mathbf{r}$ if the arguments are in the order $(\mathbf{x}_2,\mathbf{x}_1)$. If the variable is represented with three arguments, for example $T_{ijn}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}')$, the arguments to the left of the semicolon follow the same rule as described for the correlations with two spatial arguments. The argument to the right of the semicolon becomes either **r** or $-\mathbf{r}$ depending on whether $\mathbf{x}' = \mathbf{x}_1$ or $\mathbf{x}' = \mathbf{x}_2$, respectively. For the present homogeneous problem the variables are no longer functions of **X**, but only of **r**. As an example, consider the variable $T_{ijn}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_2)$, after the coordinate change and if the assumption of homogeneity is used, this correlation is denoted $T_{ijn}(\mathbf{r};-\mathbf{r})$. In all cases, the time dependence is assumed but not explicitly represented in the arguments.

B. The Correlation Equations for Homogeneous Accelerated Turbulence

The mean-velocity equation, the mean-density equation, and the mean specific volume equations are unchanged by these transformations. From symmetry $R_{ij}^{(+)}(\mathbf{r}) = R_{ji}^{(+)}(-\mathbf{r})$ and $R_{ij}^{(-)}(\mathbf{r}) = -R_{ji}^{(-)}(-\mathbf{r})$. The transformed R_{ij} equation is

$$\frac{\partial R_{ij}(r)}{\partial t} - \frac{1}{2}\left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{r};-\mathbf{r}) + \hat{H}_{ji}^{R^{(+)}}(-\mathbf{r};\mathbf{r})\right] - \frac{1}{2}\left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{r};-\mathbf{r}) - \hat{H}_{ji}^{R^{(-)}}(-\mathbf{r};\mathbf{r})\right] \tag{4.4}$$

$$+ \frac{\partial}{\partial r_n}\left[\hat{T}_{ijn}^{R^{(+)}}(\mathbf{r};-\mathbf{r}) - \hat{T}_{jin}^{R^{(+)}}(-\mathbf{r};\mathbf{r})\right]$$

$$= \frac{1}{2}\left[\Psi_{ij}^{R}(\mathbf{r}) + \Psi_{ji}^{R}(-\mathbf{r})\right] - \frac{1}{2}\left\{\left[a_i(\mathbf{r}) - \frac{m_i(\mathbf{r})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_j} + \left[a_j(-\mathbf{r}) - \frac{m_j(-\mathbf{r})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_i}\right\},$$

the $R_{ij}^{(-)}$-equation is

$$\frac{\partial R_{ij}^{(-)}(\mathbf{r})}{\partial t} - \frac{1}{2}\left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{r};-\mathbf{r}) + \hat{H}_{ji}^{R^{(-)}}(-\mathbf{r};\mathbf{r})\right] - \frac{1}{2}\left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{r};-\mathbf{r}) - \hat{H}_{ji}^{R^{(+)}}(-\mathbf{r};\mathbf{r})\right] \tag{4.5}$$

$$+ \frac{\partial}{\partial r_n}\left[2a_n R_{ij}^{(-)}(\mathbf{r}) + \hat{T}_{ijn}^{R^{(-)}}(\mathbf{r};-\mathbf{r}) - \hat{T}_{jin}^{R^{(-)}}(-\mathbf{r};\mathbf{r})\right]$$

$$= \frac{1}{2}\left[\Psi_{ij}^{R^{(-)}}(\mathbf{r}) - \Psi_{ji}^{R^{(-)}}(-\mathbf{r})\right] - \frac{1}{2}\left\{\left[a_i(\mathbf{r}) + \frac{m_i(\mathbf{r})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_j} - \left[a_j(-\mathbf{r}) + \frac{m_j(-\mathbf{r})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_i}\right\},$$

the a_i -equation is

$$\frac{\partial a_i(\mathbf{r})}{\partial t} - 2\hat{H}_i^{a}(\mathbf{r};-\mathbf{r}) + \frac{\partial}{\partial r_n}\left\{\overline{v}\left[R_{in}(\mathbf{r}) + R_{in}^{(-)}(\mathbf{r})\right] + 2a_n a_i(\mathbf{r}) + \hat{T}_{in}^{a}(\mathbf{r};\mathbf{r}) + \hat{T}_{in}^{a}(\mathbf{r};-\mathbf{r})\right\} \tag{4.6}$$

$$= -\frac{b(\mathbf{r})}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \Psi_i^{a}(-\mathbf{r}),$$

the m_i-equation is

$$\frac{\partial m_i(\mathbf{r})}{\partial t} - \hat{H}_i^m(\mathbf{r};\mathbf{r}) + \frac{\partial}{\partial r_n}\left[R_{in}^{(-)}(\mathbf{r}) - R_{in}(\mathbf{r}) + a_n m_i(\mathbf{r}) + \hat{T}_{in}^m(\mathbf{r};\mathbf{r})\right] \tag{4.7}$$

$$= b(-\mathbf{r})\frac{\partial \overline{P}}{\partial x_i} + \Psi_i^m(-\mathbf{r}),$$

and the b-equation is

$$\frac{\partial b(\mathbf{r})}{\partial t} + 2\hat{H}^b(\mathbf{r};-\mathbf{r}) + \frac{\partial}{\partial r_n}\left[a_n b(\mathbf{r}) - \bar{v} m_n(-\mathbf{r}) - a_n(\mathbf{r}) + \hat{T}_n^b(\mathbf{r};-\mathbf{r})\right] = 0. \tag{4.8}$$

C. The Fourier Transformed Correlation Equations For Homogeneous Accelerated Turbulence

Next, the equations are Fourier transformed with respect to the relative coordinate, $\mathbf{r}$:

$$\Theta(\mathbf{k},t) = \iiint_{-\infty \text{ to } +\infty} \Theta(\mathbf{r},t)e^{-ik_n r_n}d\mathbf{r}. \tag{4.9}$$

The mean-velocity equation, the mean-density equation, and the mean specific volume equations are again unchanged by these transformations. The transformed R_{ij}-equation is

$$\frac{\partial R_{ij}(\mathbf{k})}{\partial t} - \frac{1}{2}\left[\hat{H}_{ij}^{R(+)}(\mathbf{k};-\mathbf{k}) + \hat{H}_{ji}^{R(+)}(-\mathbf{k};\mathbf{k})\right] - \frac{1}{2}\left[\hat{H}_{ij}^{R(-)}(\mathbf{k};-\mathbf{k}) - \hat{H}_{ji}^{R(-)}(-\mathbf{k};\mathbf{k})\right] \tag{4.10}$$

$$+ i\mathbf{k}_n\left[\hat{T}_{ijn}^{R(+)}(\mathbf{k};-\mathbf{k}) - \hat{T}_{jin}^{R(+)}(-\mathbf{k};\mathbf{k})\right]$$

$$= \frac{1}{2}\left[\Psi_{ij}^R(\mathbf{k}) + \Psi_{ji}^R(-\mathbf{k})\right] - \frac{1}{2}\left\{\left[a_i(\mathbf{k}) - \frac{m_i(\mathbf{k})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_j} + \left[a_j(-\mathbf{k}) - \frac{m_j(-\mathbf{k})}{\overline{\rho}}\right]\frac{\partial \overline{P}}{\partial x_i}\right\},$$

the $R_{ij}^{(-)}$-equation is

$$\frac{\partial R_{ij}^{(-)}(\mathbf{k})}{\partial t}-\frac{1}{2}\left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{k};-\mathbf{k})+\hat{H}_{ji}^{R^{(-)}}(-\mathbf{k};\mathbf{k})\right]-\frac{1}{2}\left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{k};-\mathbf{k})-\hat{H}_{ji}^{R^{(+)}}(-\mathbf{k};\mathbf{k})\right] \quad (4.11)$$

$$+ik_n\left[2a_nR_{ij}^{(-)}(\mathbf{k})+\hat{T}_{ijn}^{R^{(-)}}(\mathbf{k};-\mathbf{k})-\hat{T}_{jin}^{R^{(-)}}(-\mathbf{k};\mathbf{k})\right]$$

$$=\frac{1}{2}\left[\Psi_{ij}^{R^{(-)}}(\mathbf{k})-\Psi_{ji}^{R^{(-)}}(-\mathbf{k})\right]-\frac{1}{2}\left\{\left[a_i(\mathbf{k})+\frac{m_i(\mathbf{k})}{\bar{\rho}}\right]\frac{\partial\bar{P}}{\partial x_j}-\left[a_j(-\mathbf{k})+\frac{m_j(-\mathbf{k})}{\bar{\rho}}\right]\frac{\partial\bar{P}}{\partial x_i}\right\},$$

the a_i-equation is

$$\frac{\partial a_i(\mathbf{k})}{\partial t}-2\hat{H}_i^a(\mathbf{k};-\mathbf{k})+ik_n\left\{\bar{v}\left[R_{in}(\mathbf{k})+\mathrm{R}_{\mathrm{in}}^{(-)}(\mathbf{k})\right]+2a_na_i(\mathbf{k})+\hat{\mathrm{T}}_{in}^a(\mathbf{k};\mathbf{k})\hat{\mathrm{T}}_{in}^a(\mathbf{k};-\mathbf{k})\right\} \quad (4.12)$$

$$=-\frac{b(\mathbf{k})}{\bar{\rho}}\frac{\partial\bar{P}}{\partial x_i}+\Psi_i^a(-\mathbf{k}),$$

the m_i-equation is

$$\frac{\partial m_i(\mathbf{k})}{\partial t}-\hat{H}_i^m(\mathbf{k};\mathbf{k})+ik_n\left[R_{in}^{(-)}(\mathbf{k})-R_{in}(\mathbf{k})+a_nm_i(\mathbf{k})+\hat{T}_{in}^m(\mathbf{k};\mathbf{k})\right] \quad (4.13)$$

$$=b(-\mathbf{k})\frac{\partial\bar{P}}{\partial x_i}+\Psi_i^m(-\mathbf{k}),$$

and the b-equation is

$$\frac{\partial b(\mathbf{k})}{\partial t}+2\hat{H}^b(\mathbf{k};-\mathbf{k})+ik_n\left[a_nb(\mathbf{k})-\bar{v}m_n(-\mathbf{k})-a_n(\mathbf{k})+\hat{T}_n^b(\mathbf{k};-\mathbf{k})\right]=0. \quad (4.14)$$

D. Discussion of the Fourier Transformed Equations

The Fourier transformed equations demonstrate a degree of complexity not present in the case of constant-density turbulence. However, a few general features are readily apparent, particularly in the case of isotropic turbulence. The terms in the isotropic equations fall into three categories: the turbulent dilatational terms ($\hat{H}$-terms), fluctuating pressure-, velocity-, and density-specific volume correlations (Ψ-terms), and terms multiplied by ik_n.

As noted before, the H-terms are not conservative and thus apparently represent production and destruction due to turbulent dilatation of the velocity field—these terms vanish if the instantaneous velocity field is solenoidal. It is not known under what condition these terms are "productive" or "destructive," and the direct numerical simulations of these equations currently being undertaken by D. Sandoval and J. Riley at the University of Washington will provide information about their behavior. Note that the H^b term may relate to the "dissipation" of $b(\mathbf{x})$ due to viscous diffusion, thus giving some indication of at least one correlation of this type. A similar conclusion may be readily demonstrated from the one-point correlation equations for isotropic turbulence:

$$\frac{\partial b}{\partial t} = \frac{\partial \overline{\rho}\overline{v}}{\partial t} = \overline{\rho}\frac{\partial \overline{v}}{\partial t} = 2\overline{\rho}\overline{v'\frac{\partial u_n'}{\partial x_n}} = 2\overline{\rho}\hat{H}^{\overline{v}}. \tag{4.15}$$

Hence the turbulent dilatational correlation in this case, $\hat{H}^{\overline{v}}$, represents the production and destruction of b. Note that since the trend for b in isotropic turbulence is for the materials to mix at the molecular level (if the molecular diffusivity is nonzero), one may deduce that $\hat{H}^{\overline{v}}$ is negative and represents a "destruction" or "dissipation" of b.

The Ψ-terms represent correlations with the fluctuating pressure and with the fluctuating viscous stress tensor. In constant density situations, the Ψ-term in the R_{ij}-equation is solved by the inversion of a Poisson equation and yields, for homogeneous turbulence, a "rapid term" coupling of the mean-field velocity gradient to the turbulence, and a "slow term," involving triple velocity correlations that is modeled as a tendency toward isotropy. Analogy between the constant density and variable-density cases is exploited in the variable-density single-point turbulence model (BHR) of Besnard et al. (1992) to close not only the R_{ij}-equation, but the a_i-equation as well. This approach is undoubtedly correct in the limit $\overline{(\rho' v')} \to 0$, wherein the equations approach the constant density equations exactly. However, there may be effects of large-density fluctuation that have no analogy in the constant-density case, and which therefore might be missed entirely by such an approach. Note that the b-equation does not possess a Ψ-term.

Although we will model the effects of this term by appealing to physical arguments, a more detailed discussion of other possible approaches is not within the present scope of this paper.

The terms multiplied by ik_n apparently represent a variety of effects. Such effects might include

- cascade of turbulence kinetic energy from low wavenumber/large scales to high wavenumber/small scales,
- "clumping" of heavier fluid "blobs" due to "Venturi-like" effects caused by the passage of the lighter fluid between the blobs, causing a tendency toward "blob coalescence," an inverse cascade of $b(\mathbf{k})$ and perhaps of $a_i(\mathbf{k})$ and $m_i(\mathbf{k})$, and a corresponding inhibition of forward cascade in R_{ij},
- break-up of "blobs" due to interaction of turbulent velocity fluctuations of the same size/length scales as the blobs, contributing to a forward cascade in $\mathbf{k}$-space of $b(\mathbf{k})$ and perhaps of $a_i(\mathbf{k})$ and $m_i(\mathbf{k})$.

Averaging these terms over all angles in $\mathbf{k}$-space leads directly to an ambiguity. The non-Fourier-transformed statistics are real, i.e., have a zero imaginary component. Thus the Fourier transformed correlations possess a complex conjugate symmetry:

$$\phi(\mathbf{r}) = Re[\phi(\mathbf{r})] = \phi^*(\mathbf{r}), \tag{4.16a}$$

so

$$\phi(\mathbf{k}) = \phi^*(-\mathbf{k}). \tag{4.16b}$$

Consequently, averaging over all angles in $\mathbf{k}$-space (referred to hereafter as "shell averaging"—averaging over spheres of radius $|\mathbf{k}|$ centered about the origin in $\mathbf{k}$-space) captures only the real part, but not the imaginary part, of the spectrum. Hence a shell-averaged set of equations contains only information regarding the evolution of the real parts of the spectra and must depend in some fashion on the imaginary parts for which the evolution equations contain no information. Note that this problem also occurs in the constant-density case concerning the triple-velocity correlations only, which are typically modeled. In the variable-density case, these terms include not only the higher order "triple" correlations that will be modeled, but the "lower order" correlations, which are handled with evolution equations rather than modeling. For example, consider the $b(\mathbf{k})$-equation for isotropic turbulence. Evolution of the real part of $b(\mathbf{k})$, or $b(|\mathbf{k}|)$, depends not only on the imaginary part of the "triple" correlation $T_i^b(\mathbf{k})$ but also on the imaginary parts of $a_i(\mathbf{k})$ and $m_i(\mathbf{k})$. For the simple accelerated case, the real part of $b(\mathbf{k})$ depends also on the imaginary part of $b(\mathbf{k})$. This term in the $b(\mathbf{k})$-equation for accelerated turbulence would appear to lead to simple oscillations of the b-spectrum, rather than any actual transfer of $b(\mathbf{k})$ through k-space. Such issues may be resolved when the results of the direct numerical simulations of Riley and Sandoval are available.

V. A SPECTRAL MODEL FOR VARIABLE-DENSITY TURBULENCE SUBJECTED TO ACCELERATION

A. Introduction

The model to be constructed will represent an attempt to describe the evolution of the "shell-averaged" scalar k-space statistics. As noted in the previous section, the exact, non-shell-averaged equations indicate that a substantial portion of the physics may not be adequately described by the shell-averaged quantities. Nevertheless, there may be some value in presenting a model that is at least dimensionally correct, satisfies the tensor symmetries and energy conservation, and gives a depiction of turbulence in the limit of vanishing density fluctuations that is consistent with our present understanding of turbulence of constant-density fluids. In addition, it might be argued that derivation of a more sophisticated model is unwarranted when there are so few experimental data available to verify any proposed closures. Consequently, we do not claim that our model accurately describes the evolution of the spectra of variable-density turbulence; in part, because we do not know how turbulence with large density variations evolves.

We have derived evolution equations for five spectral quantities: $R_{ij}^{(+)}$, $R_{ij}^{(-)}$, a_i, m_i, and b. Note that the Fourier spectrum of $R_{ij}^{(-)}$ must be entirely imaginary, and since we desire a "shell-averaged" model that incorporates only the real parts of the spectra, we have dropped the $R_{ij}^{(-)}$-equation. Next, note that the single-point statistics of a_i and m_i must be related to each other by a simple factor of $-\bar{\rho}$. The single-point statistics are equal to the scalar-k-space statistics integrated over all k. Thus we have

$$\int_0^\infty a_i(k,t)dk = -\frac{1}{\bar{\rho}}\int_0^\infty m_i(k,t)dk. \tag{5.1}$$

This integral equality does not imply a "spectral" equality: $m_i(k,t) = -\bar{\rho}\, a_i(k,t)$. Thus, even though an evolution equation for $m_i(k,t)$ might conceivably give spectra that do not satisfy a spectral equality, it must satisfy the above integral inequality. Note, however, that an initially stationary variable-density mixture subjected to a sudden pressure gradient the m_i and a_i-equations appear to indicate that

$$\frac{\partial a_i(\mathbf{k})}{\partial t} \cong -\frac{b(\mathbf{k})}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i}, \tag{5.2}$$

and

$$\frac{\partial m_i(\mathbf{k})}{\partial t} \cong b(-\mathbf{k}) \frac{\partial \overline{P}}{\partial x_i} = b^*(\mathbf{k}) \frac{\partial \overline{P}}{\partial x_i}, \tag{5.3}$$

from which one might reasonably deduce that

$$m_i(\mathbf{k}, t) = -\overline{\rho} a_i^*(\mathbf{k}, t), \tag{5.3}$$

indicating that the shell-averaged quantities may satisfy a spectral equality, at least to this limit. Due to a lack of more detailed knowledge regarding the spectral behavior of the turbulence, and also realizing that we have perhaps already made a tremendous concession to physical fidelity by describing the physics in a scalar-k-space, we will assume the spectral equality

$$m_i(k, t) = -\overline{\rho} a_i(k, t). \tag{5.4}$$

Further, in the absence of molecular effects (viscosity and diffusion), we will require energy conservation. This requires that

$$\frac{\partial}{\partial t}\left[\tilde{U}_n(t)\tilde{U}_n(t) + R_{nn}(t)\right] = 0, \tag{5.5}$$

which for the simple accelerated turbulence reduces to

$$\frac{\partial}{\partial t}\left[\int_0^\infty a_n(q,t)dq \int_0^\infty a_n(s,t)ds + \int_0^\infty R_{nn}(k,t)dk\right] = 0. \tag{5.6}$$

Finally, in the limit of vanishing b, we require that a_i (and thus m_i) also vanish and that the turbulence will behave like constant-density turbulence with a passive scalar.

B. Modified Spectral Equations for Homogeneous Accelerated Turbulence

If we assume spectral equality between m_i and a_i, then

$$m_i(\mathbf{k},t) = -\overline{\rho} a_i^*(\mathbf{k},t) = -\overline{\rho} a_i(-\mathbf{k},t), \tag{5.7}$$

the spectral equations may be simplified as follows: the R_{ij}-equation becomes

$$\frac{\partial R_{ij}(\mathbf{k})}{\partial t} - \frac{1}{2}\left\{\left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{k};-\mathbf{k}) + \hat{H}_{ji}^{R^{(+)}}(-\mathbf{k};\mathbf{k})\right] + \left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{k};-\mathbf{k}) - \hat{H}_{ji}^{R^{(-)}}(-\mathbf{k};\mathbf{k})\right]\right\} \tag{5.8}$$

$$+ ik_n\left[\hat{T}_{ijn}^{R^{(+)}}(\mathbf{k};-\mathbf{k}) - \hat{T}_{jin}^{R^{(+)}}(-\mathbf{k};\mathbf{k})\right]$$

$$= Re\left[\Psi_{ij}^{R}(\mathbf{k})\right] - \left\{Re\left[a_i(\mathbf{k})\right]\frac{\partial \overline{P}}{\partial x_j} + Re\left[a_j(\mathbf{k})\right]\frac{\partial \overline{P}}{\partial x_i}\right\},$$

the $R^{(-)}$-equation becomes

$$\frac{\partial R_{ij}^{(-)}(\mathbf{k})}{\partial t} - \frac{1}{2}\left\{\left[\hat{H}_{ij}^{R^{(-)}}(\mathbf{k};-\mathbf{k}) + \hat{H}_{ji}^{R^{(-)}}(-\mathbf{k};\mathbf{k})\right] + \left[\hat{H}_{ij}^{R^{(+)}}(\mathbf{k};-\mathbf{k}) - \hat{H}_{ji}^{R^{(+)}}(-\mathbf{k};\mathbf{k})\right]\right\} \tag{5.9}$$

$$+ 2ik_n a_n R_{ij}^{(-)}(\mathbf{k}) + ik_n\left[\hat{T}_{ijn}^{R^{(-)}}(\mathbf{k};-\mathbf{k}) - \hat{T}_{jin}^{R^{(-)}}(-\mathbf{k};\mathbf{k})\right]$$

$$= Im\left[\Psi_{ij}^{R^{(-)}}(\mathbf{k})\right] - \left\{Im\left[a_i(\mathbf{k})\right]\frac{\partial \overline{P}}{\partial x_j} + Im\left[a_j(\mathbf{k})\right]\frac{\partial \overline{P}}{\partial x_i}\right\},$$

the a_i-equation is unchanged,

$$\frac{\partial a_i(\mathbf{k})}{\partial t} - 2\hat{H}_i^a(\mathbf{k};-\mathbf{k}) + ik_n\left\{\overline{v}\left[R_{in}(\mathbf{k}) + R_{in}^{(-)}(\mathbf{k})\right] + 2a_n a_i(\mathbf{k}) + \hat{T}_{in}^a(\mathbf{k};\mathbf{k}) + \hat{T}_{in}^a(\mathbf{k};-\mathbf{k})\right\} \tag{5.10}$$

$$= -\frac{b(\mathbf{k})}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \Psi_i^a(-\mathbf{k}),$$

and the b-equation becomes

$$\frac{\partial b(\mathbf{k})}{\partial t} + 2\hat{H}^b(\mathbf{k};-\mathbf{k}) + ik_n\left[a_n b(\mathbf{k}) + ba_n(\mathbf{k})\right] + ik_n\hat{T}_n^b(\mathbf{k};-\mathbf{k}) = 0. \tag{5.11}$$

C. Modeling

As stated previously, we attempt to construct model equations that describe the evolution of the shell-averaged spectra of the turbulence quantities. Obviously, the shell average of $R_{ij}^{(-)}$ is zero, and hence this quantity is of no use for present purposes. This leaves three turbulence quantities for which to construct equations: $R_{ij}(k,t)$, $a_i(k,t)$, and $b(k,t)$. We will require the model to satisfy the following constraints:

1. If b vanishes, so must a_i.
2. If $R_{\alpha n}$ vanishes, so must a_α.
3. In the limit of vanishing b, the turbulence must behave as a constant density turbulence with a passive scalar.

We will assume that transfer in scalar-k-space can be described by nonlinear advection and diffusion, as was assumed in the diffusion approximation of Leith (1967) and in the constant-density spectral model (BHRZ) of Besnard et al. (1990). Note that this model of Besnard et al. (1990), when extended to the isotropic decay of a passive scalar, does not yield the so-called Batchelor k^{-1} scaling range for the passive scalar at high Prandtl number. This is clearly due to the fact that for the case of finite turbulent Reynolds number, the turbulence time scale used in the BHRZ cascade model goes to zero at high wavenumbers, and therefore does not permit the passive scalar to be"advected" past the turbulence energy dissipation range. Therefore, an integral time scale for the turbulence cascades will be used, rather than the nonintegral ("local") form used by Besnard et al. (1990).

D. The R_{ij} Model Equation

First, we begin with the R_{ij}-equation. As noted previously, the $\hat{H}$-terms represent production or destruction of turbulence due to the dilatation of the instantaneous fluctuating velocity field. The dilatation of the velocity field in our case will be due to molecular diffusivity, as indicated in Section 2, Eq. (2.7), by

$$\frac{\partial u_n}{\partial x_n} = -\frac{\partial}{\partial x_n}\left(\frac{D}{\rho}\frac{\partial \rho}{\partial x_n}\right).$$

It thus seems reasonable to assume that these terms will scale in some sense like other molecular effects, e.g., viscous dissipation or molecular diffusion. These terms must have the dimensions of [(mass)^1(length)^3(time)$^{-3}$]. As the density fluctuations vanish or as the molecular diffusivity vanishes, the instantaneous velocity field becomes solenoidal, and these terms must also vanish. Thus a candidate model for these terms is

$$+ C_{RH}\beta_{RH}(k,t)\overline{D}k^2R_{ij}(k,t), \tag{5.12}$$

where β_{RH} is a dimensionless function of *b(k,t)* , which, for simplicity, we will choose to be

$$\beta_{RH}(k,t) = kb(k,t). \tag{5.13}$$

(There are, in fact, an infinity of possibilities for β-terms.) We are currently ignorant as to whether these terms represent production or destruction (or perhaps both); hence, the ambiguity of sign. C_{RH} is a dimensionless model constant, presumably with an order of magnitude of one.

Next, consider the terms explicitly multiplied by ik_n. The term that also involves triple correlations is identical to terms arising in constant-density turbulence and is said to represent turbulence cascade in k-space. These terms are modeled by Besnard et al. (1990) as an advection and diffusion in k-space. We will use an advection/diffusion model in k-space but will also include the possibility of an alteration of cascade due to the presence of density fluctuations and will use an integral form for the time scale:

$$- C_{R2}\frac{\partial}{\partial k}\left\{\left[1+C_{RB}\beta_{RB}(k,t)\right]\left[\frac{1}{\overline{\rho}}\int_0^k q^2R_{nn}(q,t)dq\right]^{1/2}\left[\frac{C_{R1}}{C_{R2}}kR_{ij}(k,t)-k^2\frac{\partial R_{ij}(k,t)}{\partial k}\right]\right\}, \tag{5.14}$$

where again the β is a dimensionless function of *b(k,t)* and represents an alteration of cascade due to the presence of density fluctuations. For simplicity, one may choose the same β as before:

$$\beta_{RB}(k,t) = kb(k,t), \tag{5.15}$$

thus, in the limit as b vanishes, the cascade terms become identical to those proposed by Besnard et al. (1990).

Next, we consider the fluctuating pressure part of the viscous stress correlations. A simple Poisson equation cannot be derived for the fluctuating pressure in the same sense that one may be derived for constant-density turbulence. If the instantaneous velocity field is assumed to be solenoidal, than a type of Poisson equation can be derived [See Besnard et al. (1990)]. However, it is not apparent that this Poisson-like equation can be inverted (i.e., solved) via a Green's function operator as is done for the constant-density case. We also note that even in the case of constant-density turbulence, the solution of the Poisson equation provides guidance only for modeling of the pressure-velocity correlations, rather than being an actual term for use in the model [see, for example, Besnard et al. (1990)]. Thus for present purposes, we will base our model of the effects of the pressure-velocity correlations on dimensional consistency, physical arguments, and analogy with the constant-density case. First, we note that the model for the fluctuating pressure-velocity correlations must approach the form for constant-density turbulence in the limit of vanishing b. Thus, the model should include a "slow part," reflecting a tendency toward isotropy and, in the more general case, a "rapid part" that couples the spectral tensor to mean-flow velocity gradients. Because there are no mean-flow velocity gradients in our homogeneous accelerated turbulence and our isotropic turbulence, we may at present neglect the rapid part. The tendency toward isotropy is modeled in direct analogy to the constant-density case:

$$- C_{RTI}\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[R_{ij}(k,t)-\frac{1}{3}\delta_{ij}R_{nn}(k,t)\right]. \tag{5.16}$$

Next, we consider additional effects that may arise due to the presence of density fluctuations. We expect that the interpenetration of one fluid into the other results in a net production of Reynolds stress, and this net production is also reflected as a net decrease in a_i due to drag effects. The effects leading to drag/decay of a_i are more easily intuited than is the consequent production of R_{ij}, and thus we will first discuss the modeling of the drag/decay of a_i arising from fluctuating pressure correlations and then assure that the corresponding term in R_{ij} is consistent and satisfies energy conservation. For the model of the fluctuating pressure correlations of the a_i-equation we will assume the form

$$- \left\{C_{RP1}\beta_{RP1}(k,t)k^2\sqrt{a_n(k,t)a_n(k,t)} + C_{RP2}\beta_{RP2}(k,t)\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\right\}a_i(k,t). \tag{5.17}$$

The β-terms are again dimensionless functions of b and the C_{RP1} and C_{RP2} are dimensionless model constants. Note that the left-hand term in the []-braces represents the wavenumber-dependent time scales for the drag of a_i due to the spectral distribution of the interpenetrating fluids. The right-hand term in the braces represents a time scale for the destruction of a_i due to a disorganization, or isotropization, caused by the presence of the turbulence. As noted previously, in the absence of molecular effects and external forcing, energy conservation requires that

$$\frac{\partial}{\partial t}\left[\int_0^\infty R_{nn}(k,t)dk + \int_0^\infty a_n(k,t)dk \int_0^\infty a_n(k,t)dk\right] = 0. \tag{5.18}$$

Thus energy conservation requires that the production term in the R_{ij}-equation must have an integral form. The form we will assume is

$$\begin{aligned} &+ \bar{\rho}\left[C_{RP1}\beta_{RP1}(k,t)k^2\sqrt{a_n(k,t)a_n(k,t)}\right]\left[a_i(k,t)a_j(t) + a_j(k,t)a_i(t)\right] \\ &+ \bar{\rho}\left\{C_{RP2}\beta_{RP2}(k,t)\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\right\}\left[a_i(k,t)a_j(t) + a_j(k,t)a_i(t)\right]. \end{aligned} \tag{5.19}$$

The coupling of a_i with the mean-pressure gradient need not be modeled. One purpose of these β-terms is to ensure that a_i vanishes as b vanishes. The form for both these terms is chosen as

$$\beta_{RP1}(k,t) = \beta_{RP2}(k,t) = \frac{1}{b(t)}, \tag{5.20}$$

thus, as $b(t)$ vanishes, the drag term becomes infinite, forcing a_i to vanish. We do not anticipate that this will cause an infinite production in the R_{ij}-equation because a_i appears quadratically in the production term in the R_{ij}-equation and will vanish in the limit as b vanishes, thus forcing the production term to zero. We also do not argue that these β-terms are correct or that they are the only reasonable choice. They appear to be, however, the simplest forms that satisfy both the dimensional requirements and also provide a consistency between the vanishing of b and the vanishing of a_i. The fluctuating viscous stress tensor will also be modeled by direct analogy to the constant-density case:

$$-2\frac{\bar{\mu}}{\bar{\rho}}k^2 R_{ij}(k,t), \tag{5.21}$$

where fluctuations in the molecular viscosity are neglected and any apparent stresses arising from the dilatation of the turbulent field are assumed to be taken into account by the modeling of the $\hat{H}$-terms. The model for the evolution of $R_{ij}(k,t)$ is, therefore,

$$\frac{\partial R_{ij}(k,t)}{\partial t} = -C_{R2}\frac{\partial}{\partial k}\left\{\left[1+C_{RB}\beta_{RB}(k,t)\right]\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[\frac{C_{R1}}{C_{R2}}kR_{ij}(k,t)-k^2\frac{\partial R_{ij}(k,t)}{\partial k}\right]\right\}$$

$$+\left[C_{RP1}\beta_{RP1}(k,t)\bar{\rho}k^2\sqrt{a_n(k,t)a_n(k,t)}\right]\left[a_i(k,t)a_j(t)+a_j(k,t)a_i(t)\right]$$

$$+\left\{C_{RP2}\beta_{RP2}(k,t)\bar{\rho}\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\right\}\left[a_i(k,t)a_j(t)+a_j(k,t)a_i(t)\right]$$

$$-\left[a_i(k,t)\frac{\partial\bar{P}}{\partial x_j}+a_j(k,t)\frac{\partial\bar{P}}{\partial x_i}\right]$$

$$-C_{RTI}\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[R_{ij}(k,t)-\frac{1}{3}\delta_{ij}R_{nn}(k,t)\right]-\left[2\frac{\bar{\mu}}{\bar{\rho}}-\bar{D}C_{RH}\beta_{RH}(k,t)\right]k^2R_{ij}(k,t). \quad (5.22)$$

E. The a_i Model Equation

The modeling of the a_i-equation follows from the same assumptions used in the R_{ij} modeling. The $\hat{H}^a$-term is modeled as a dissipation term (molecular diffusion of density causes a decrease in a_i), and the viscous dissipation is modeled in the same fashion as for the R_{ij}-equation:

$$-\left[\frac{\bar{\mu}}{\bar{\rho}}+C_{aD}\beta_{aD}(k,t)\bar{D}\right]k^2a_i(k,t). \quad (5.23)$$

The terms explicitly by ik_n are assumed to represent transfer in k-space. Note that the cascade terms may include time scales from the so-called triple correlations: the spectral tensor components, spectral a_i, and the spectrally integrated a_i. In addition, if we assume that a "clumping," or agglomeration, of interpenetrating fluid "blobs" may occur due to "Venturi-like" effects of flow between smaller clumps, it seems reasonable that this effect will be present in both $b(k,t)$ and $a_i(k,t)$ and may be described as an inverse cascade in k-space. This assumption is justified by the fact that no net increase in the quantity of interpenetrating material is given and only the length scales associated with the correlation between density fluctuations may be growing. In addition, we anticipate that as the density fluctuations become vanishingly small, the density fluctuations will behave in a manner analogous to a passive scalar, the dominant transport in k-space will be toward higher wavenumbers, and the time scales for transfer will be identical to the time scales for transport of the turbulence energy itself, which we will here assume is given by the model of Besnard et al. (1990). Three different time scales (or k-space velocity scales) will be used to describe the advection and diffusion in k-space:

$$
-C_{aR2}\frac{\partial}{\partial k}\left\{\left[1+C_{aB}\beta_{aB}(k,t)\right]\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[\frac{C_{aR1}}{C_{aR2}}ka_i(k,t)-k^2\frac{\partial a_i(k,t)}{\partial k}\right]\right\} \quad (5.24)
$$

$$
-C_{aa2}\frac{\partial}{\partial k}\left\{k^3\sqrt{a_n(k,t)a_n(k,t)}\left[\frac{C_{aa1}}{C_{aa2}}a_i(k,t)-k\frac{\partial a_i(k,t)}{\partial k}\right]\right\}
$$

$$
+C_{aA2}\sqrt{a_n(t)a_n(t)}\frac{\partial}{\partial k}\left\{k^2\left[\frac{C_{aA1}}{C_{aA2}}a_i(k,t)+k\frac{\partial a_i(k,t)}{\partial k}\right]\right\}.
$$

The first line is turbulence transport of a_i in k-space due to the turbulent velocity field. The second line describes the transfer due to length- and scale-dependent interpenetration and mixing. The third line describes clumping and "inverse" transfer to large scales/small wavenumbers due to the Venturi-like effects caused by interpenetration of the ensemble (i.e., the spectrally integrated time scale) of fluids. The choice of βs will be deferred until the discussion of the $b(k,t)$-equation.

The fluctuating pressure correlations are modeled as drag and are discussed in previous subsection, Eq. (5.17). We will repeat them here for convenience:

$$
-\left\{C_{RP1}\beta_{RP1}(k,t)k^2\sqrt{a_n(k,t)a_n(k,t)}+C_{RP2}\beta_{RP2}(k,t)\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\right\}a_i(k,t).
$$

Note that the fluctuating pressure correlations for the $R_{ij}(k,t)$-equation also include a return to isotropy[3]. However, for our vector $a_i(k,t)$, a return to isotropy is equivalent to a decay of a_i and is thus physically analogous to a decay and will be assumed to be included in this drag term. The coupling of $b(k,t)$ to the mean-pressure gradient does not need to be modeled. The model equation for $a_i(k,t)$ is thus

$$\frac{\partial a_i(k,t)}{\partial t} = -C_{aR2}\frac{\partial}{\partial k}\left\{\left[1+C_{aB}\beta_{aB}(k,t)\right]\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[\frac{C_{aR1}}{C_{aR2}}ka_i(k,t)-k^2\frac{\partial a_i(k,t)}{\partial k}\right]\right\}$$

$$-\ C_{aa2}\frac{\partial}{\partial k}\left\{k^3\sqrt{a_n(k,t)a_n(k,t)}\left[\frac{C_{aa1}}{C_{aa2}}a_i(k,t)-k\frac{\partial a_i(k,t)}{\partial k}\right]\right\}$$

$$+\ C_{aA2}\sqrt{a_n(t)a_n(t)}\frac{\partial}{\partial k}\left\{k^2\left[\frac{C_{aA1}}{C_{aA2}}a_i(k,t)+k\frac{\partial a_i(k,t)}{\partial k}\right]\right\}$$

$$-\left\{C_{RP1}\beta_{RP1}(k,t)k^2\sqrt{a_n(k,t)a_n(k,t)}+C_{RP2}\beta_{RP2}(k,t)\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\right\}a_i(k,t)$$

$$-\ \frac{b(k,t)}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i}-\left[\frac{\bar{\mu}}{\bar{\rho}}+2C_{aD}\beta_{aD}(k,t)\bar{D}\right]k^2 a_i(k,t). \tag{5.25}$$

F. The b Model Equation

The modeling of the b-equation follows the same rationale as does the modeling of the a_i-equation. Again we assume the $\hat{H}^b$-term represents a molecular diffusive dissipation of b due to turbulent dilatation. The rationale for this assumption was discussed in Section II.F. Thus the $\hat{H}^b$-term is modeled as

$$-\ 2\bar{D}k^2 b(k,t). \tag{5.26}$$

---

[3]Recent direct numerical simulations by Sandoval, Riley and Clark have also identified a "rapid-term" in the fluctuating pressure-specific volume correlation that responds immediately to the mean pressure gradient. This is analogous to the so-called "rapid-term" in Reynolds stress-Epsilon models that couples mean flow gradients directly to the Reynolds stresses, but is neglected in the above model.

The transfers in k-space are assumed to arise from the terms explicitly multiplied by ik_n and will be modeled in direct analogy to the transfer terms for the a_i-equation. The b-equation is simply

$$\frac{\partial b(k,t)}{\partial t} = -C_{bR2}\frac{\partial}{\partial k}\left\{\left[1 + C_{bB}\beta_{bB}(k,t)\right]\left[\frac{1}{\bar{\rho}}\int_0^k q^2 R_{nn}(q,t)dq\right]^{1/2}\left[\frac{C_{bR1}}{C_{bR2}}kb(k,t) - k^2\frac{\partial b(k,t)}{\partial k}\right]\right\}$$

$$- C_{ba2}\frac{\partial}{\partial k}\left\{k^3\sqrt{a_n(k,t)a_n(k,t)}\left[\frac{C_{ba1}}{C_{ba2}}b(k,t) - k\frac{\partial b(k,t)}{\partial k}\right]\right\}$$

$$+ C_{bA2}\sqrt{a_n(t)a_n(t)}\frac{\partial}{\partial k}\left\{k^2\left[\frac{C_{bA1}}{C_{bA2}}b(k,t) + k\frac{\partial b(k,t)}{\partial k}\right]\right\} - 2\bar{D}k^2 b(k,t). \quad (5.27)$$

Note that if the mixing fluids are immiscible, then by definition the mixing fluids never diffuse into each other due to molecular effects. In the immiscible case the interspersed fluid may form "droplets" of a small but finite size wherein the surface tension effects counterbalance the turbulent shearing. In this case the cascading to smaller scales due to breakup of fluid blobs will cease, and the interspersed fluid is never diffused into the surrounding fluid due to molecular effects. Likewise, if we assume that one of the mixing fluids is in fact a particulate field (e.g., fluidized beds) the "fluid" lumps can be broken up no further than the size of a particle. In both of these cases, the density fluctuations as represented by $b(t)$ are conserved in a homogeneous field, even if the turbulent Reynolds number is infinite. The density fluctuations, b, cannot be conserved by simply setting the diffusivity to zero because at infinite Reynolds number, b will be cascaded to infinity where the distinction between "molecular mixing" and "chunk mixing" is irrelevant. In other words, the density fluctuations will be dissipated at infinite Reynolds number in the same manner as the velocity fluctuations are dissipated unless the cascade terms for $b(k,t)$, and consequently $a_i(k,t)$, are modified in some fashion. Thus a simple model for β_{bB} is proposed that permits $b(t)$ to be conserved at infinite Reynolds number by negating the turbulent cascade at high wavenumbers. Thus in this case, β_{bB} represents a spectral "block" at high wavenumbers,

$$\beta_{bB}(k) = exp\left(-\left[kD_p\right]^{n_p}\right) - \frac{1}{C_{bB}},$$

where D_p represents a length scale (say, a measure of particulate or droplet size) and n_p is an exponent to be selected. Of course, this model for β_{bB} is simply heuristic, but will permit $b(k,t)$ to be cascaded to smaller scales due to turbulence and will also permit it to be conserved. Note that fluids that diffuse at the molecular level are described by setting D_p to zero. A similar choice is made for β_{aB} in the $a_i(k,t)$-equation.

G. Discussion

A model has been proposed for the evolution of the shell-averaged spectra of variable-density turbulence subjected to an acceleration. The model is constructed from arguments of dimensional and tensorial consistency with the unmodeled equations, from heuristic arguments regarding the effects of large-density fluctuations, and by analogy with constant-density turbulence. The model possesses many terms and many unknown model constants, reflecting both the complexity of the problem of modeling variable density turbulence and the uncertainty of our knowledge of the problem. Some coefficients may be determined by comparison to the BHRZ model of Besnard et al. (1990) (see also Clark, 1992). However, other coefficients must be determined by comparison with experimental data, which in most cases are not adequate for determination of the spectral behavior of the turbulence, or by comparison with direct numerical simulations of variable-density turbulence, such as those currently being undertaken by Sandoval and Riley at the University of Washington. Until these results are available, the choice of model coefficients must be considered tentative, guided largely by intuition and reason.

H. Sample Computations with the Spectral Model

The cases to be computed are of homogeneous turbulence subjected a body force, i.e., an acceleration. The model equations, in abbreviated notation, are

$$\frac{\partial R_{nn}(k,t)}{\partial t} = \{\text{Production of } R_{nn}\} + \left\{\begin{matrix} k\text{-space} \\ \text{transfer of } R_{nn} \end{matrix}\right\} - 2a_1(k,t)\frac{\partial \overline{P}(t)}{\partial x_1}, \tag{5.28}$$

$$\frac{\partial a_1(k,t)}{\partial t} = -\{\text{Drag of } a_1\} + \left\{\begin{matrix} k\text{-space} \\ \text{transfer of } a_1 \end{matrix}\right\} - \frac{b(k,t)}{\overline{\rho}}\frac{\partial \overline{P}(t)}{\partial x_1}, \tag{5.29}$$

and

$$\frac{\partial b(k,t)}{\partial t} = \left\{\begin{matrix} k\text{-space} \\ \text{transfer} \end{matrix}\right\}. \tag{5.30}$$

The mean velocity is given by

$$\frac{\partial \tilde{U}_1(t)}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{P}(t)}{\partial x_1} + g_1(t). \tag{5.31}$$

The acceleration, g_1, is imposed and the mean-pressure gradient is computed by noting that, by definition,

$$\frac{\partial}{\partial t}\left[\tilde{U}_1(t) + \int_0^\infty a_1(k,t)dk\right] = 0, \tag{5.32}$$

giving

$$\frac{\partial \bar{P}}{\partial x_1} = \frac{\bar{\rho} g_1(t) + \bar{\rho}\int_0^\infty \left[\left\{\begin{matrix} k\text{-space} \\ \text{transfer of } a_1\end{matrix}\right\} - \{\text{Drag of } a_1\}\right]dk}{1 + \int_0^\infty b(k,t)dk}. \tag{5.33}$$

Note that in the absence of density fluctuations, this expression for the mean-pressure gradient reduces to the hydrostatic case:

$$\frac{\partial \bar{P}}{\partial x_1} = \bar{\rho} g_1(t). \tag{5.34}$$

The model constants are shown in Table I. Some of the coefficients have been chosen by comparison with the BHRZ spectral model (Clark 1992). Other coefficients are unknown and must be determined by comparison with relevant experiments or direct numerical solutions of the governing equations. Thus the values presented for these coefficients must be regarded for now as being provisional.

The computations are for turbulence subjected to abrupt changes in acceleration and must be considered as illustrative until suitable data are available to verify the model and coefficients. The first three cases represent a miscible mixture of two fluids with a Schmidt number of Sch = 1.0. All three cases share the same initial conditions, which is a quiescent fluid (R_{ij} and a_i = 0) with an initial distribution of density fluctuations for which $b(k,t) = b_0(k/k_0)^2 e^{-(k/k_0)^2}$, where $k_0 = 1$, and b_0 is chosen so that $b(t_0) = 1.0$. In Case 1 the fluid is subjected to a constant acceleration, $g = 1$, for a total time of $t = 30$. In Case 2, this quiescent fluid is subjected to an acceleration $g = 1$ for a time of $t = 4$, and then allowed to freely decay. In Case 3, the fluid is subjected to an acceleration of $g = 1$ for a time of $t = 4$, and then subjected to a acceleration reversal, $g = -1$. In all three cases the molecular diffusivity is 10^{-3}, the viscosity is 10^{-3}, and the initial value of $b(t)$ is 1. Note that, in the following discussion, spectrally integrated (one-point) values will be distinguished from the spectral values by the absence of the wavenumber in the arguments. Also note that $K_{nn}(t)$ is one-half the value of $R_{nn}(k,t)$ integrated over all k .

Figures 1 through 4 show the evolution of the density-specific volume correlation, $b(t)$, the turbulent mass flux, $a_x(t)$, the turbulent kinetic energy, $K_{nn}(t)$, and the mean-pressure gradient, respectively, for all three cases. Examination of the data indicates that at very early times ($t < 2$) $b(t)$ is essentially constant, $K_{nn}(t)$ is growing quadratically, and $a_l(t)$ is growing linearly in time. At intermediate times ($t \approx 2$) $b(t)$ is beginning to decay, $K_{nn}(t)$ is growing at an approximately linear rate in time, and $a_1(t)$ is growing at a less than linear rate in time, reflecting the process of spectral equilibration. At time of approximately $t \approx 3$, $a_x(t)$ actually begins to decay, even though the acceleration is continuing, apparently because of the decreased coupling with pressure due to the decrease in $b(t)$ as well as the effects of the drag terms. For later times during the constant acceleration (Case 1), the $K_{nn}(t)$ reaches a maximum at about $t = 5$ and begins to decay in response to the drop-off of $a_x(t)$ and turbulent dissipation. The $a_x(t)$ and $b(k,t)$ continue to decay. For the case where the acceleration is "turned off" at $t = 4$ (Case 2), the turbulence kinetic energy and turbulent mass flux begin to decay immediately. Note that for Case 2, $b(t)$ actually decays somewhat more slowly than it does for the continuously accelerated case, apparently due to the decrease the turbulent cascade to the dissipation region, as reflected by the smaller value of $K_{nn}(t)$ in Case 2 as compared with Case 1 (see Figure 3). Where the acceleration is reversed (Case 3), the turbulent kinetic energy is significantly suppressed after the reversal and the subsequent growth is small, due to the small response of $a_x(t)$, which is attributable to the small values of $b(t)$ available to couple with the pressure gradient to drive $a_x(t)$.

Figures 5 through 7 show the evolution of spectra of $b(k,t)$, $a_1(k,t)$, and $R_{nn}(k,t)$, respectively, for Case 1. Note that initially only the $b(k,t)$ is nonzero. At early times ($t \approx 0.1$) the spectra for $a_x(k,t)$ and $R_{nn}(k,t)$ are driven directly by the coupling between the mean-pressure gradient and $b(k,t)$ [for $a_x(k,t)$] and $ax(k,t)$ [for $R_{nn}(k,t)$]; hence, the spectra look qualitatively like the initial $b(k,t)$ spectrum. As time increases, an inertial range develops in the region from $0 \leq \log_e(k) \leq 4$. For $b(k,t)$ and $R_{nn}(k,t)$, the inertial range follows the typical Kolmogorov (1941) scaling of $k^{-5/3}$. However, the spectrum for $a_x(k,t)$ approximately scales as $k^{-11/5}$, somewhat steeper than b or R_{nn} , apparently due to the influence of drag.

Figures 8 through 10 show the evolution of the spectra during free decay at $t > 4$, subsequent to the acceleration of $g = 1$ for $t \leq 4$. The spectra of $b(k,t)$ and $R_{nn}(k,t)$ show an approximately self-similar decay of the high wavenumber spectra, and the low wavenumbers appear to be invariant in time. Note that the spectrum of $a_x(k,t)$ shows a "collapse" of spectrum at $\log_e(k,) \geq -1$. This is apparently a manifestation of the influence of the "destruction" of a_x due to drag, wherein the production of a_x due to coupling of $b(k,t)$ with the mean-pressure field is no longer active. The result of the "destruction" of the high-wavenumber part of the spectrum due to drag is that a "peaked" spectral form for a_x arises (the peak at $t = 24$ is located at $\log_e(k) \approx -1.5$).

Figures 11 through 13 show the evolution of the spectra after a reversal of the acceleration at $t = 24$. As shown in Figure 11, $b(k,t)$ continues to decay in an approximately self-similar fashion. Figure 12 indicates that $a_x(k,t)$ changes sign at the higher wavenumbers before changing sign at the lower wavenumbers. The more rapid behavior of the high wavenumbers of a_x is due to the combined influence of the drag (which tends to drive the high wavenumbers toward zero faster than it does the low wavenumbers) and of the change in sign of the mean-pressure gradient, which, when coupled to b, drives a_x to change sign. Note that the relatively slow response of the low-wavenumber part of the a_x spectrum combined with the change in sign of the mean-pressure gradient causes a suppression of the large scales (low wavenumbers) of $R_{nn}(k,t)$, as shown in Figure 13 at $t \approx 8.0$. However, after $a_x(k,t)$ fully changes sign, $R_{nn}(k,t)$ resumes its growth in an approximately self-similar fashion.

Figures 14 through 16 show scaled spectra at $t = 24.0$ for the three cases. [Note that $k_{b,\max}$ is the value of k where $b(k,t)$ is a maximum. Likewise, at $k = k_{a,\max}$ the value of $a_x\,(k,t)$ is external; and at $k = k_{R,\max}$, the value of $R_{nn}(k,t)$ is a maximum.] The purpose of these figures is to compare the shapes of the spectra that have evolved for each of the three cases. Figure 14 shows a scaled plot of $b(k,t)$ for the three cases. This figure shows that in all three cases, the spectra look similar at wavenumbers below the dissipation region ($\log_e(\mathrm{k}) \approx 6$). The same is true of $R_{nn}(k,t)$, as shown in Figure 16. Note that the spectrum of $a_x(k,t)$ during free decay (Case 2) is strikingly different from the spectra of the two accelerated cases (Cases 1 and 3), as shown in Figure 15. The reasons for the suppression of the high-wavenumber part of the a_x-spectrum during free decay were discussed above.

Cases 4 and 5 utilize the spectral "block" at high wavenumbers to mimic the possible consequences of a surface tension between two fluids or of the presence of particles wherein the collective "cloud" of particles constitutes one of the fluids. For these cases, the viscosity of the fluid is the same as that of the first three cases, the molecular diffusivity is zero, and the spectral block is placed at $\log_e(k) = 6.0$. In Case 4 the fluid is subjected to a continuous acceleration of $g = 1$. In Case 5 the acceleration is "bumped" up from $g = 1$ to $g = 10$ at time $t = 10.0$.

Figure 17 shows the evolution of $b(t)$ for Cases 4 and 5. Note that $b(t)$ is conserved for both cases because the molecular diffusivity is zero and the spectral "block" prevents $b(k,t)$ from being cascaded to infinity. Figure 18 shows the evolution of $a_x(t)$ for Cases 4 and 5. In both cases, $a_x(t)$ increases in response to the acceleration, reaches an extremum, and then diminishes. Throughout this time, $b(t)$ remains constant, and the mean-pressure gradients grow to approximately constant values (see Figure 19). The cause of the subsequent decreases in the magnitudes of $a_x(t)$ is apparently due to the fact that the spectrum of $b(k,t)$ is being moved to higher wavenumbers where it is "captured" by the spectral block in a region where viscous effects dominate the behavior of $a_x(t)$ (this is demonstrated in Figures 20 and 21). Hence the coupling of $b(k,t)$ with the mean-pressure gradients to produce $a_x(k,t)$ is overwhelmed by the viscous effects, drag effects, and turbulence cascade effects on $a_x(k,t)$. Similarly the bump in acceleration to $g = 10.0$ at $t = 10.0$ does not produce as big an increase in $a_x(t)$ as did the initial acceleration of $g = 1.0$ because by $t = 10.0$, $b(k,t)$ has moved to wavenumbers where it is less effective at producing $a_x(k,t)$.

Figure 22 shows the evolution of the turbulent kinetic energy for Cases 4 and 5. In contrast to the behavior of $a_x(t)$, $K_{nn}(t)$ responds dramatically to the bump in acceleration at $t = 10.0$. $R_{nn}(k,t)$ is driven by the coupling of the mean-pressure gradient to $a_x(k,t)$, which has not moved to high wavenumbers as has the spectrum of $b(k,t)$. Thus, although the pressure coupling to $b(k,t)$ is overwhelmed by viscosity, drag, and turbulence cascade, the pressure coupling with $a_x(k,t)$ is not likewise overwhelmed. Thus $K_{nn}(t)$ responds much more dramatically to the acceleration than did $a_x(t)$.

Figures 23 through 25 show the evolution of the spectra $b(k,t)$, $a_x(k,t)$, and $R_{nn}(k,t)$, respectively, for Case 4. Figures 20, 21, and 26 show these spectra for Case 5 for $t \geq 10.0$. The important feature to note in these graphs is the distribution of $b(k,t)$ at the inverse particle size, $\log_e(k) = 6.0$. As seen in Figures 20 and 23, a local maximum of $b(k,t)$ is formed at the inverse particle size scale, whereas for $a_x(k,t)$ (Figures 21 and 24) and $R_{nn}(k,t)$ (Figures 25 and 26), a "knee" or "kink" is formed in the spectrum at this point. Note the these are log-log plots, and thus a significant portion of $b(k,t)$ has accumulated at the particle size scale. However, for both $a_x(k,t)$ and $R_{nn}(k,t)$, the dominant scale is clearly at wavenumbers that are small compared with the particle size.

VI. CONCLUSIONS

The two-point correlation equations for variable-density incompressible turbulence, represented in mass-averaged variables, have been derived. For the restrictive case of homogeneous turbulence-subjected acceleration, Fourier transformed equations have been used with respect to the separation distance between the two points. Several necessary constraints on the correlations have been derived, and a spectral closure in scalar-k-space which satisfies these constraints has been postulated. The model reduces to the BHRZ spectral turbulence model for the case of constant-density turbulence. Computations with the model produced conditions wherein departures from spectral self-similarity of the model spectrum occur due to changes in the acceleration driving the turbulence. These departures from self-similarity were also manifested in variations in the ratio of the turbulence length scales for the turbulent kinetic energy, the turbulent mass flux, and the fluctuating density correlations and represent phenomena that one-point engineering models are ill-equipped to capture due to their implicit assumptions regarding the relationships of the length scales and time scales of the turbulence.

The proposed spectral model has not been subjected to rigorous comparisons with experimental data or direct numerical simulations, and thus the model's predictions should be viewed with some caution. However, as a conceptual tool, the model provides a more general description of the physics of turbulence with large-density gradients than is provided by the one-point engineering closures. The spectral model may thus permit the researcher to study the consequences of the restrictions implicit in one-point closures, as well as to formulate more general one-point descriptions of variable-density turbulence. Future work with the model should include both rigorous testing of the model against appropriate experimental results and direct numerical simulations of variable-density turbulence and include as well analytical and numerical investigations of the model. The analytical and numerical investigations may include various schemes for reducing the model to a one-point model under various assumptions and restrictions in order to study the behavior and adequacy of these one-point models as compared with existing one-point closures.

ACKNOWLEDGMENTS

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TABLE I

A. R_{ij} Model Constants

Coefficient	Value	Rationale
C_{R1}	0.121212	Equipartition, Kolmogorov const.
C_{R2}	0.060606	Equipartition, Kolmogorov const.
C_{RP1}	0.5	Provisional
C_{RP2}	0.5	Provisional
$C_{R\Pi}$	0.17508	BHRZ coefficient C_M
C_{RB}	0	Preliminary DNS results
C_{RH}	1.0	Provisional

B. a_i Model Constants

Coefficient	Value	Rationale
C_{aR1}	0.121212	Passive scalar—related to R_{ij}
C_{aR2}	0.060606	Passive Scalar—related to R_{ij}
C_{aa1}	–0.121212	Provisional—inverse cascade
C_{aa2}	0.060606	Provisional
C_{aA1}	0.0	Provisional
C_{aA2}	0.0,	Provisional
C_{RP1}	1.0	Determined for R_{ij} model
C_{RP2}	1.0	Determined for R_{ij} model
C_{RB}	0.0	Preliminary DNS results
C_{aD}	1.0	Provisional

C. b Model Constants

Coefficient	Value	Rationale
C_{bR1}	0.121212	Passive scalar—related to R_{ij}
C_{bR2}	0.060606	Passive scalar—related to R_{ij}
C_{ba1}	–0.121212	Provisional—inverse cascade
C_{ba2}	0.060606	Provisional
C_{bA1}	0.0	Provisional
C_{bA2}	0.0	Provisional
C_{RB}	0.0	Preliminary DNS results

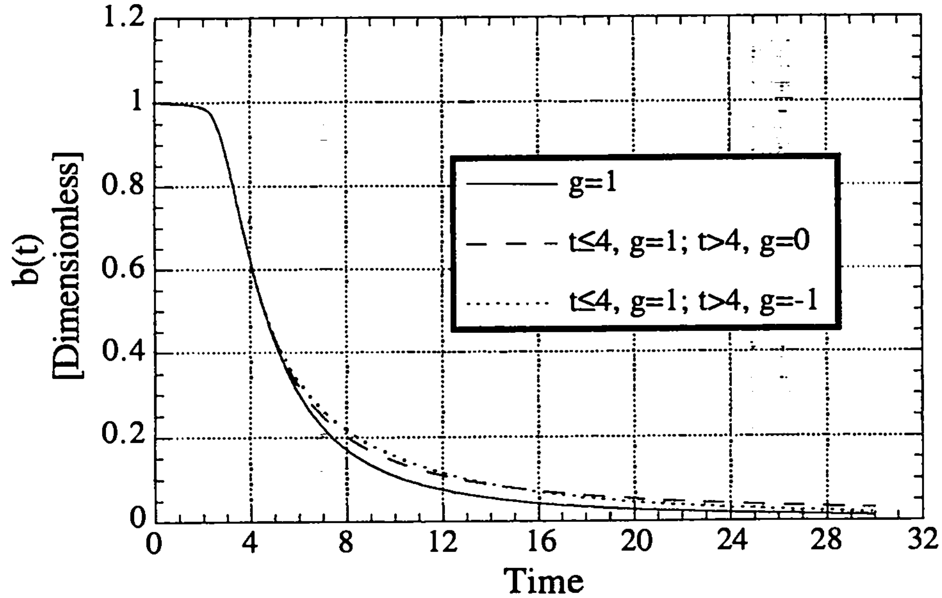

Figure 1. Evolution of b(t) versus time for Cases 1 through 3.

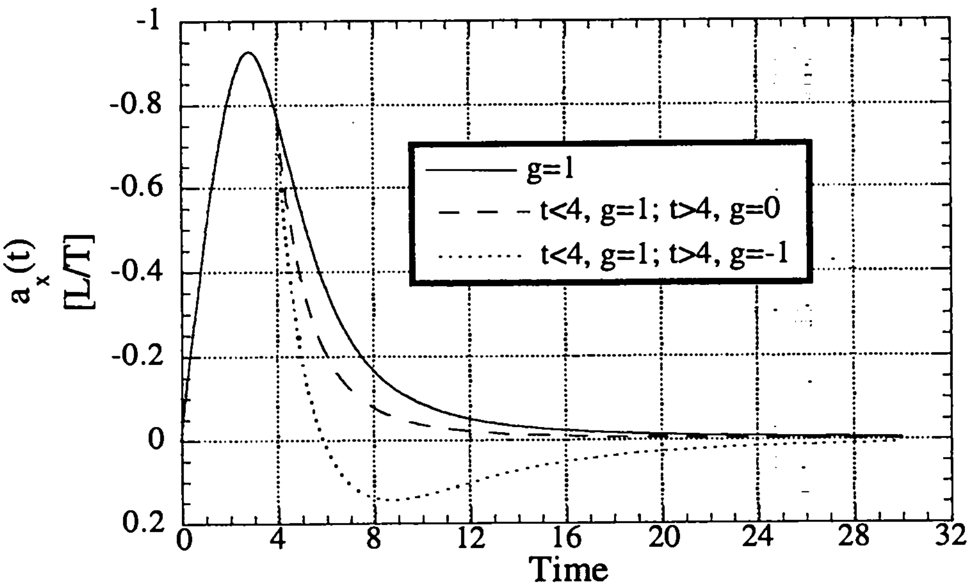

Figure 2. $a_x(t)$ versus time for Cases 1 through 3.

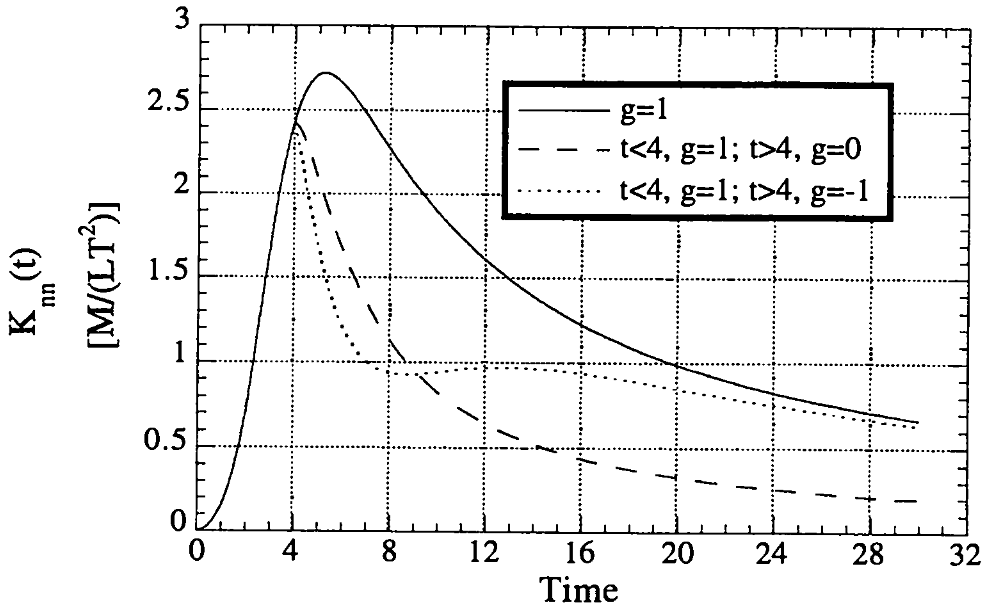

Figure 3. $K_{nn}(t)$ versus time for Cases 1 through 3.

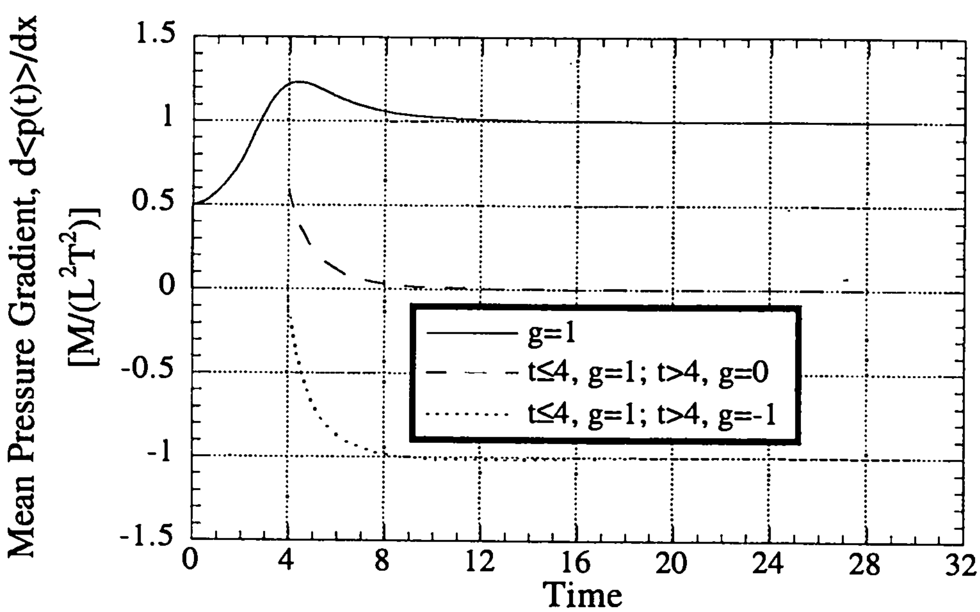

Figure 4. Mean pressure gradient versus time for Cases 1 through 3.

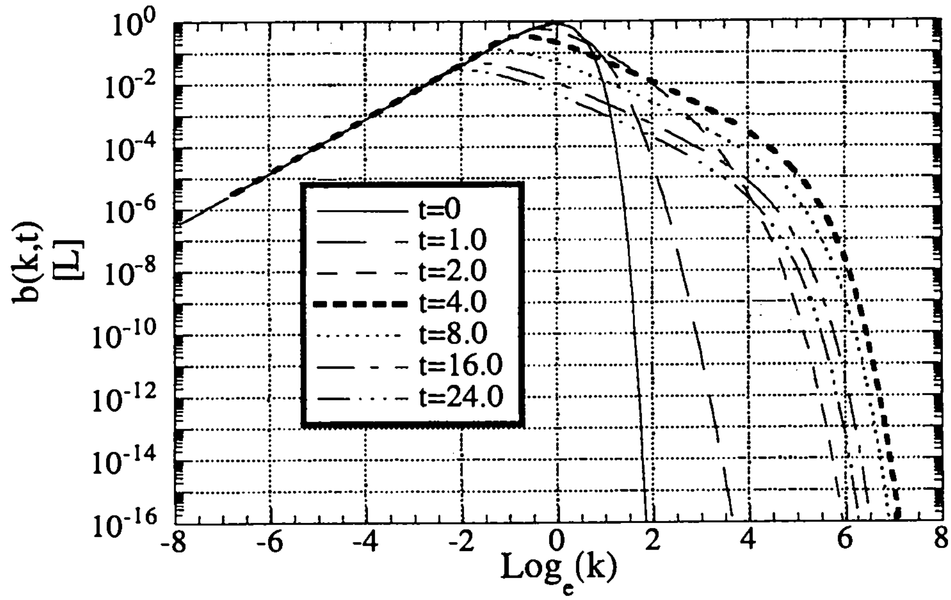

Figure 5. Evolution of the spectrum of b(k,t) during continuous acceleration (Case 1).

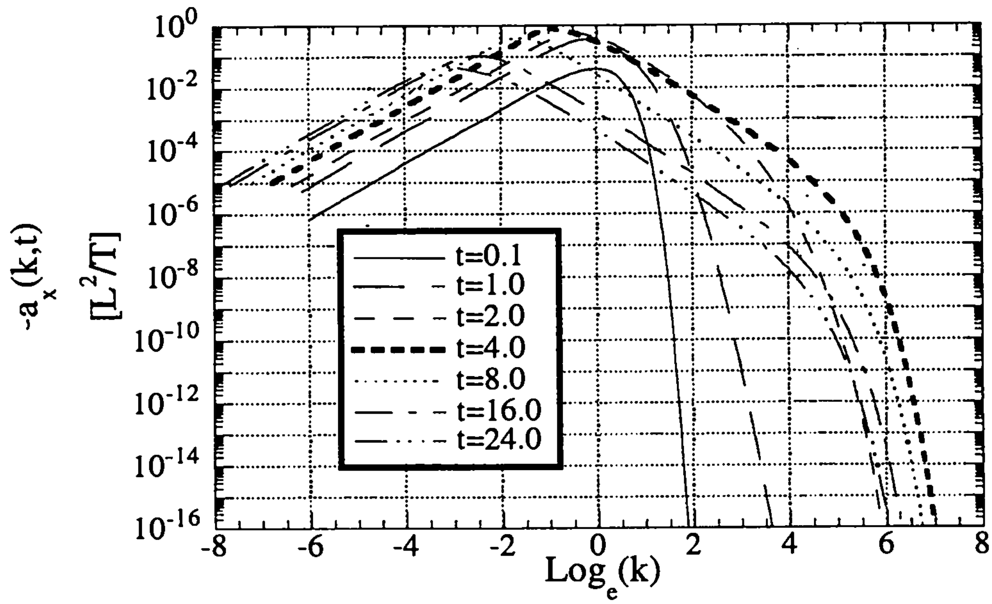

Figure 6. Evolution of the spectrum of $a_x(k,t)$ during continuous acceleration (Case 1).

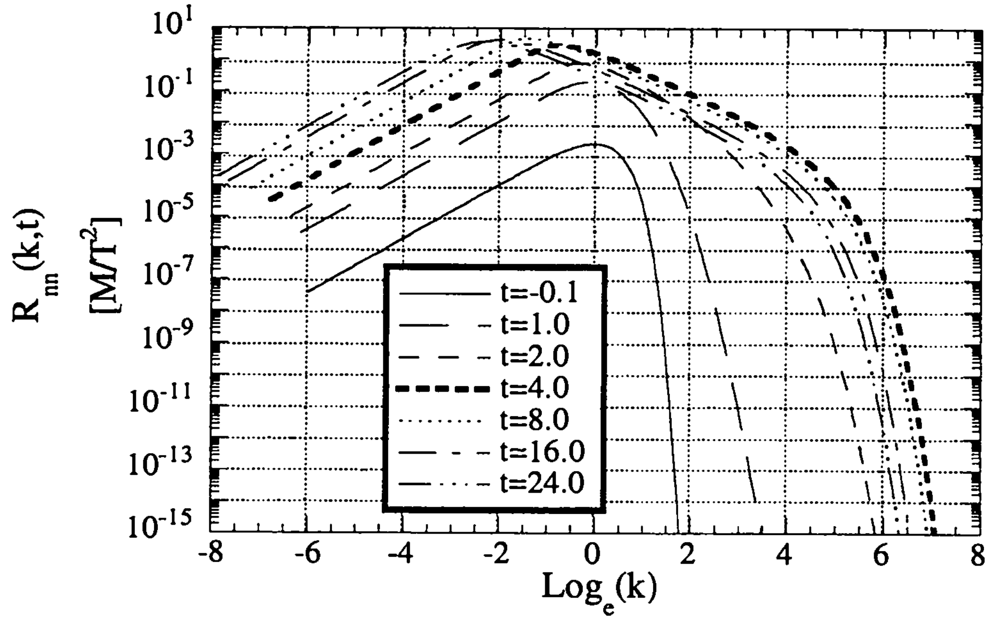

Figure 7. Evolution of the spectrum the $R_{nn}(k,t)$ during continuous acceleration (Case 1).

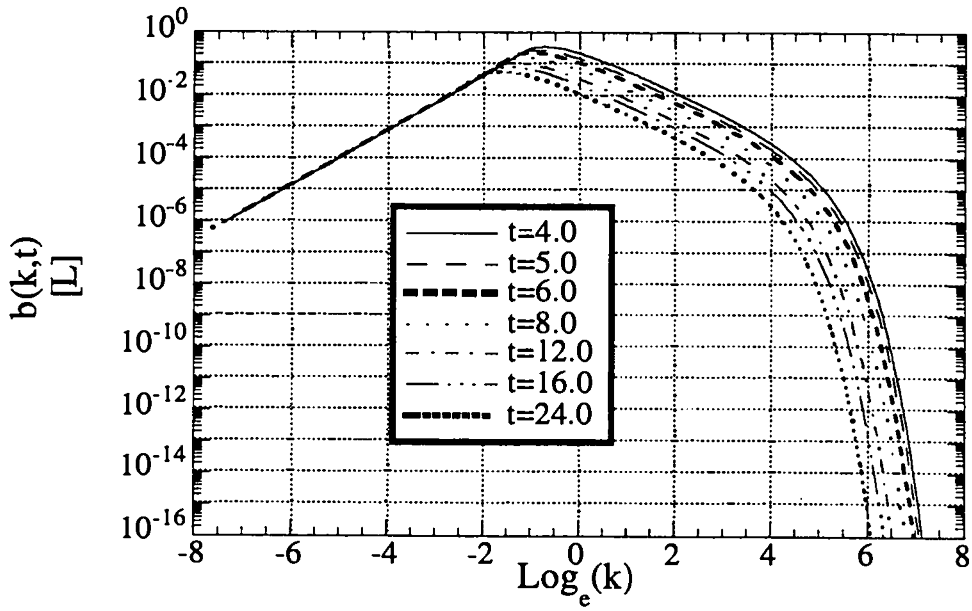

Figure 8. Evolution of the spectrum of b(k,t) during free decay after initial acceleration (Case 2).

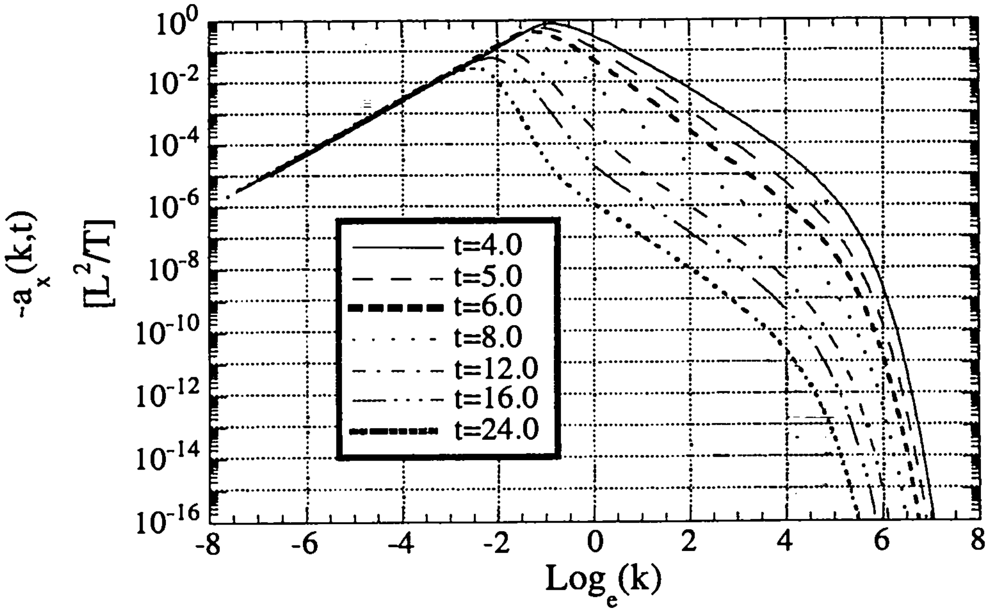

Figure 9. Evolution of the spectrum of $a_x(k,t)$ during free decay after initial acceleration (Case 2).

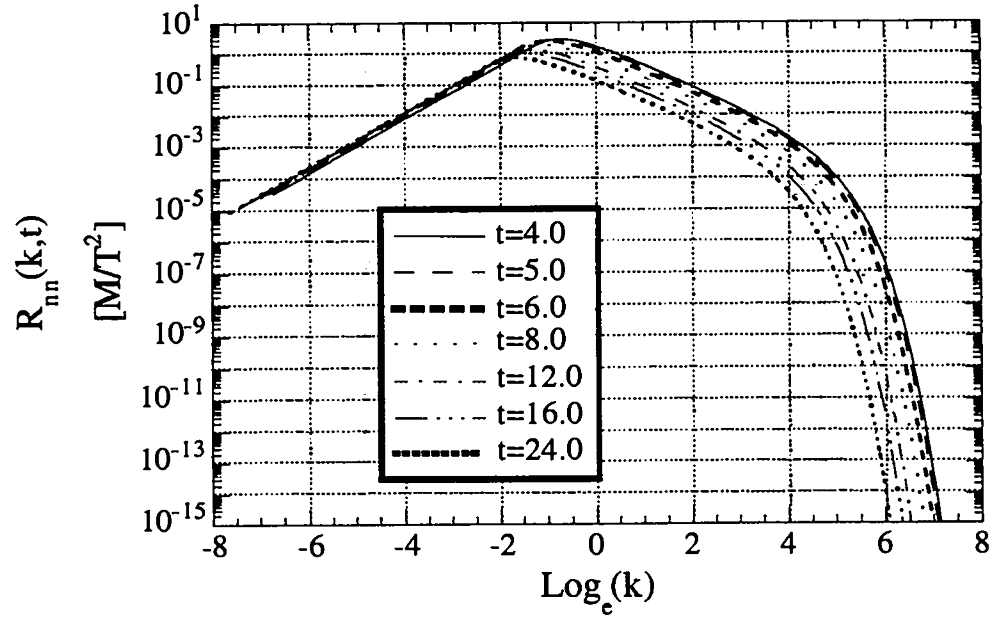

Figure 10. Evolution of the spectrum of $R_{nn}(k,t)$ during free decay after initial acceleration (Case 2).

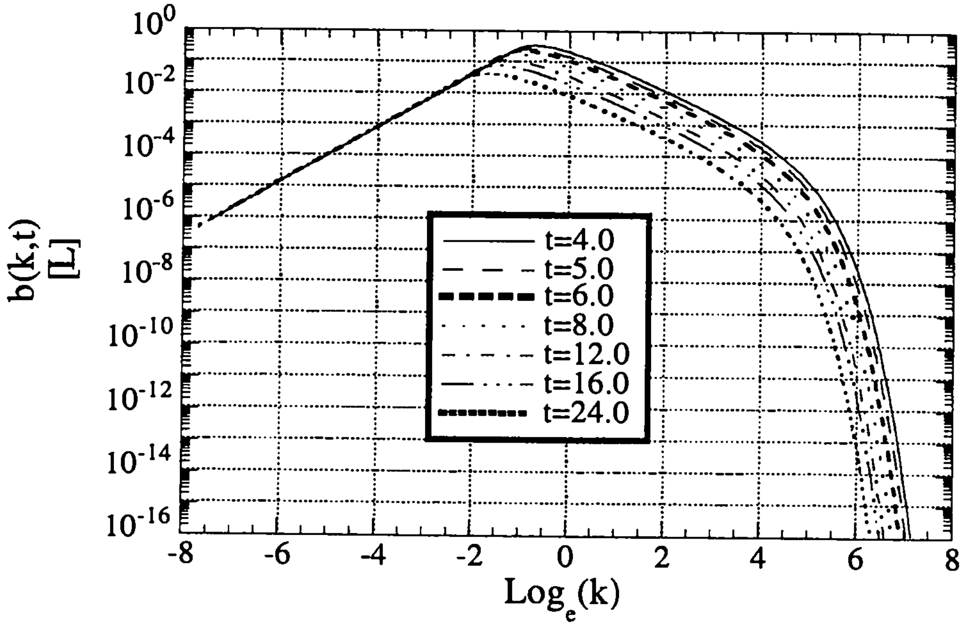

Figure 11. Evolution of the spectrum of b(k,t) after acceleration reversal (Case 3).

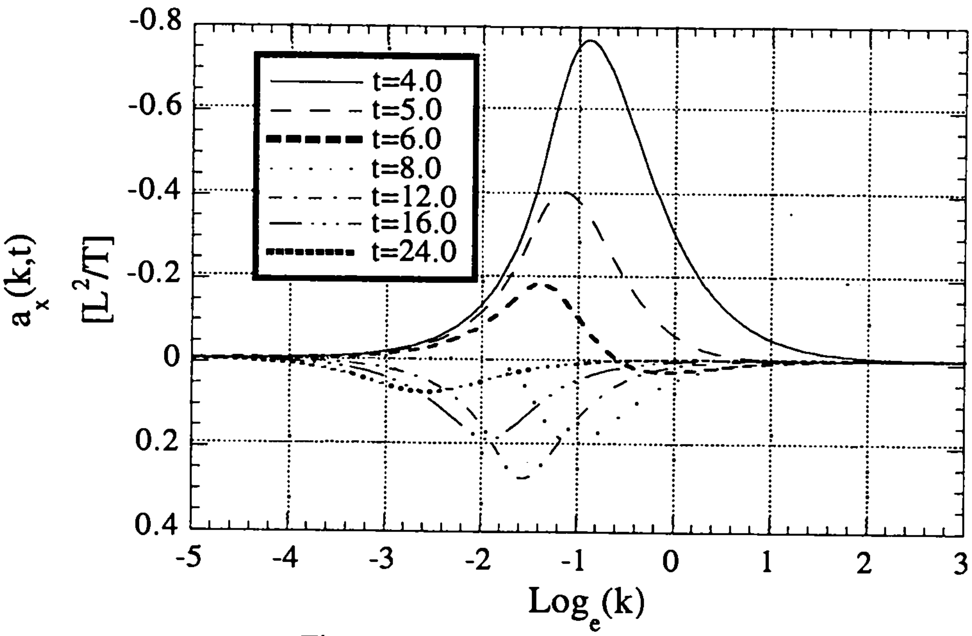

Figure 12. Evolution of the spectrum of a(k,t) after acceleration reversal (Case 3).

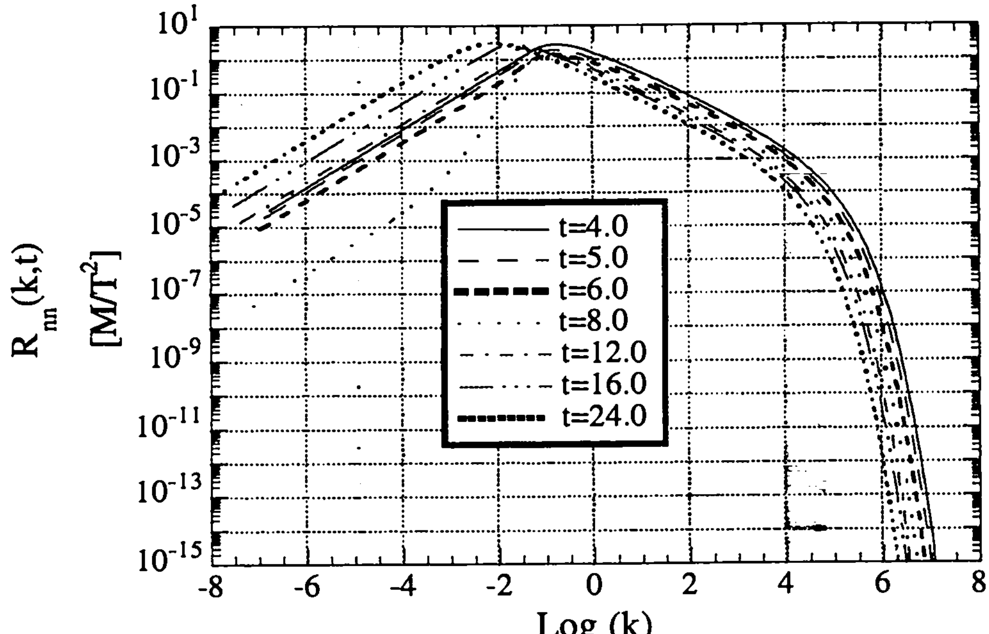

Figure 13. Evolution of the Spectrum of $R_{nn}(k,t)$ after acceleration reversal (Case 3).

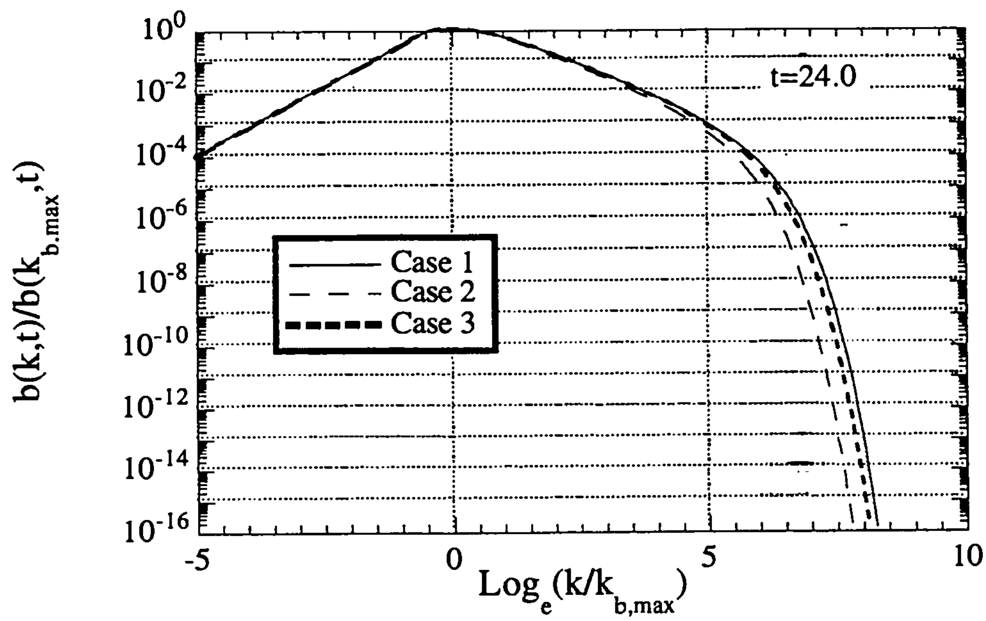

Figure 14. Scaled plots of the b-spectrum for three cases:
1. Continuous Acceleration, 2. Acceleration/Free Decay, and
3. Reversed Acceleration.

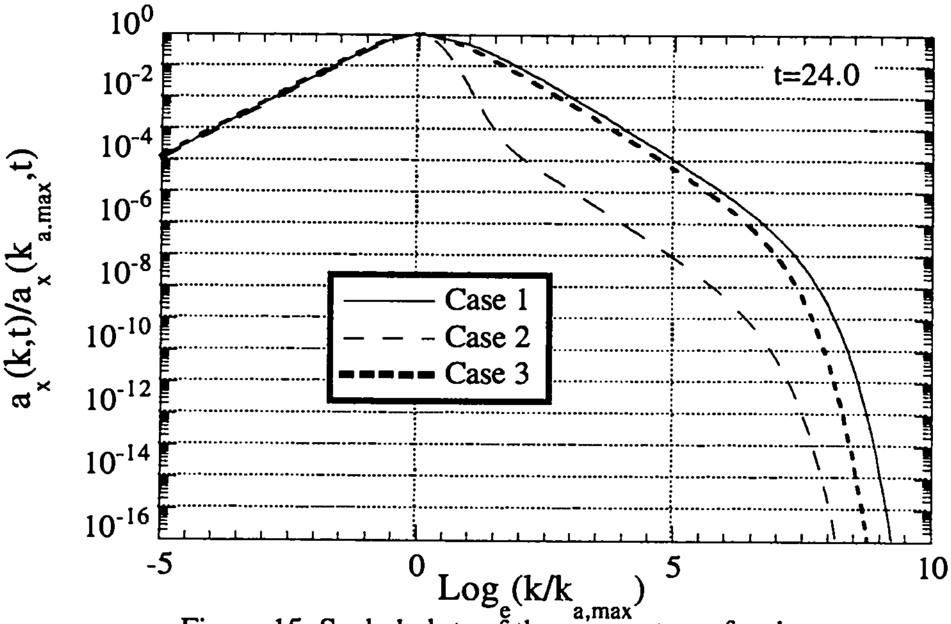

Figure 15. Scaled plots of the a_x-spectrum for three cases:
1. Continuous Acceleration, 2. Acceleration/Free Decay, and
3. Reversed Acceleration.

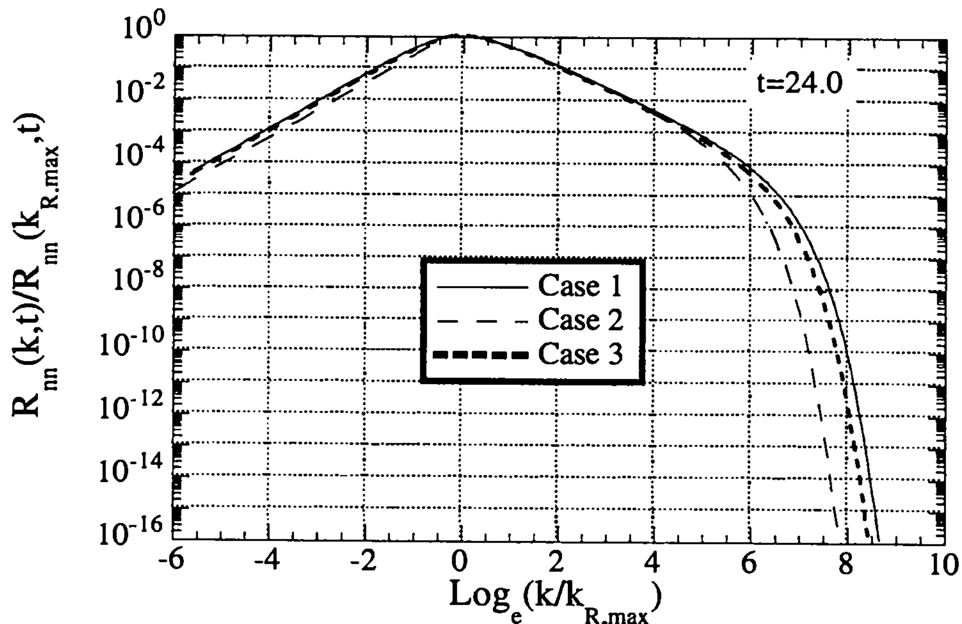

Figure 16. Scaled plots of the R_{nn}-spectrum for three cases:
1. Continuous Acceleration, 2. Acceleration/Free Decay, and
3. Reversed Acceleration.

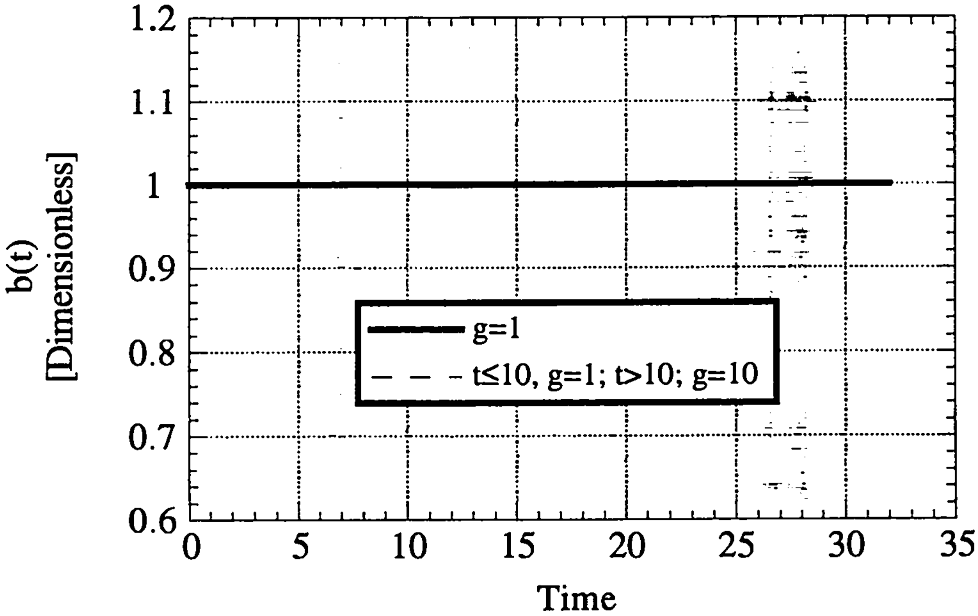

Figure 17. b(t) versus time for Cases 4 and 5.

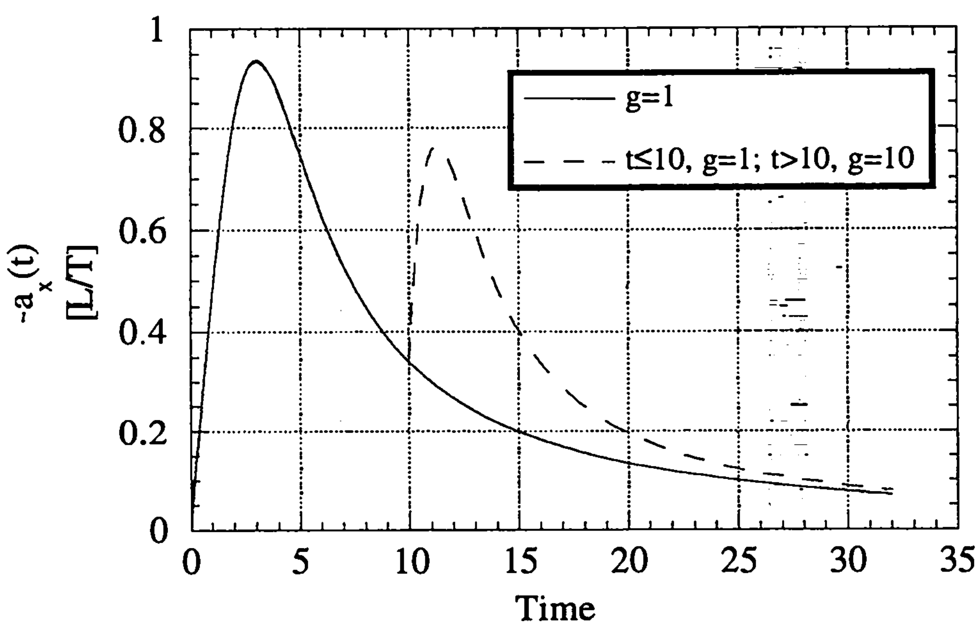

Figure 18. $a_x(t)$ versus time for Cases 4 and 5.

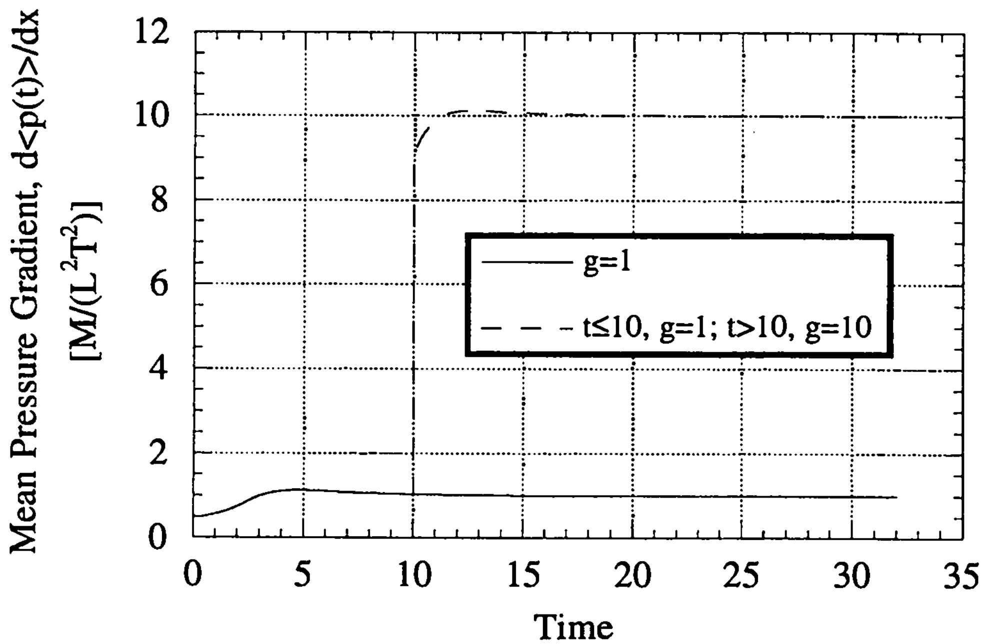

Figure 19. Mean pressure gradient versus time for Cases 4 and 5.

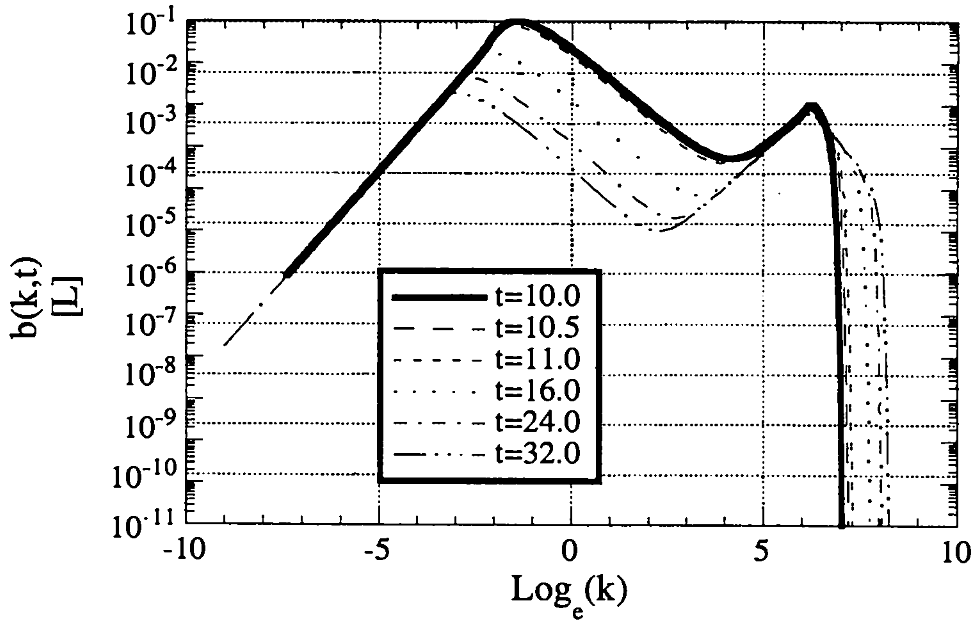

Figure 20. Evolution of the spectrum of b(k,t) after increase in acceleration (Case 5).

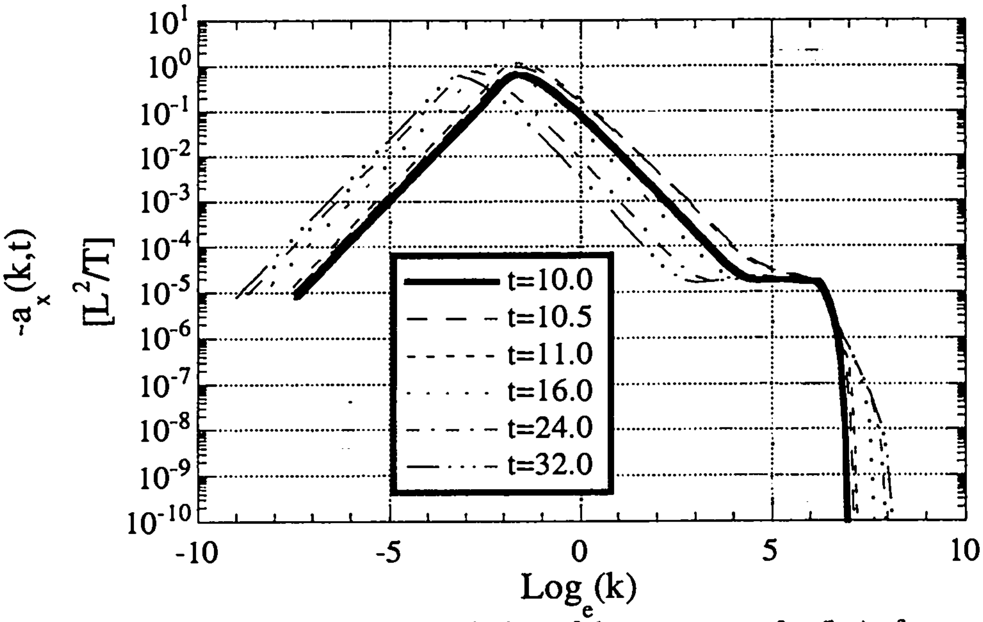

Figure 21. Evolution of the spectrum of $a_x(k,t)$ after an increase in the acceleration (Case 5).

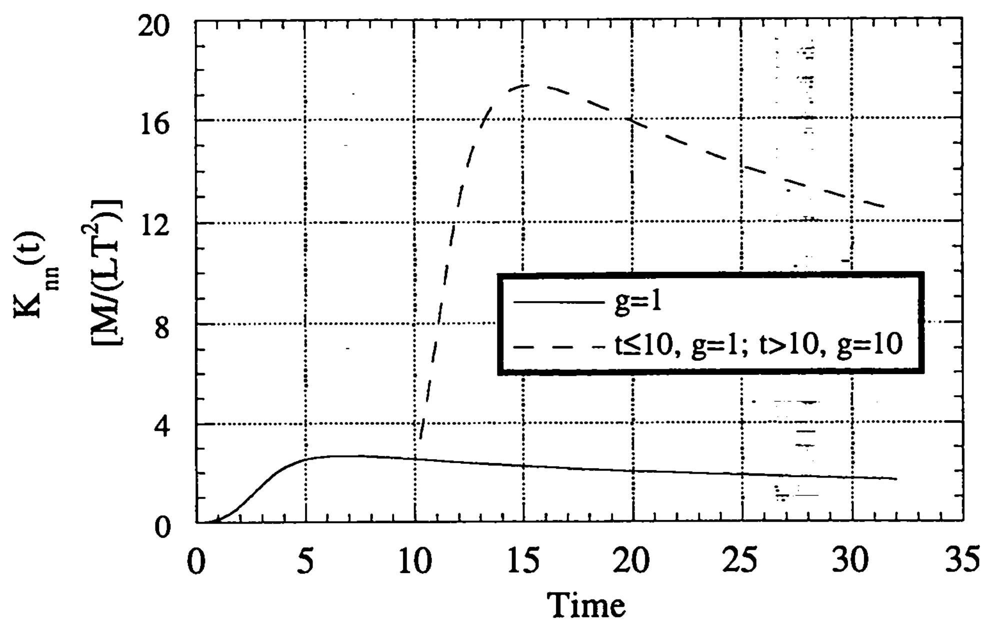

Figure 22. $K_{nn}(t)$ versus time for Cases 4 and 5.

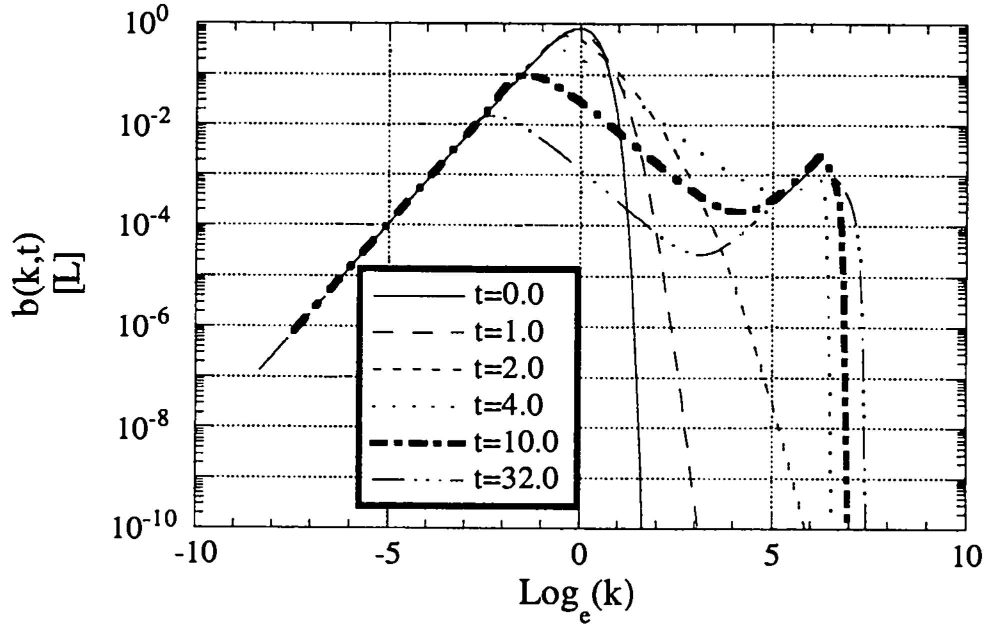

Figure 23. Evolution of the spectrum of b(k,t) during constant acceleration (Case 4).

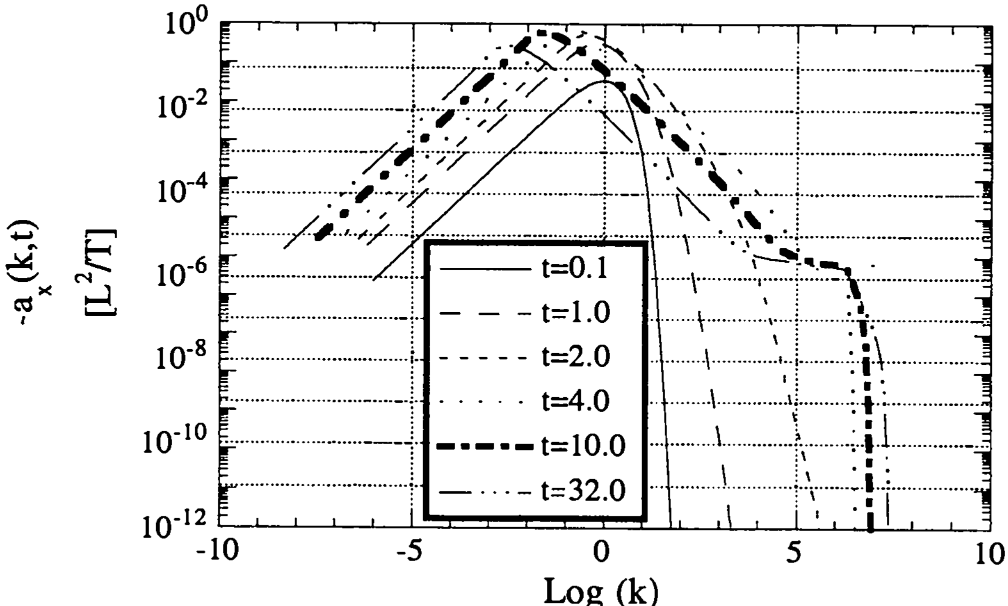

Figure 24. Evolution of the spectrum of $a_x(k,t)$ during constant acceleration (Case 4).

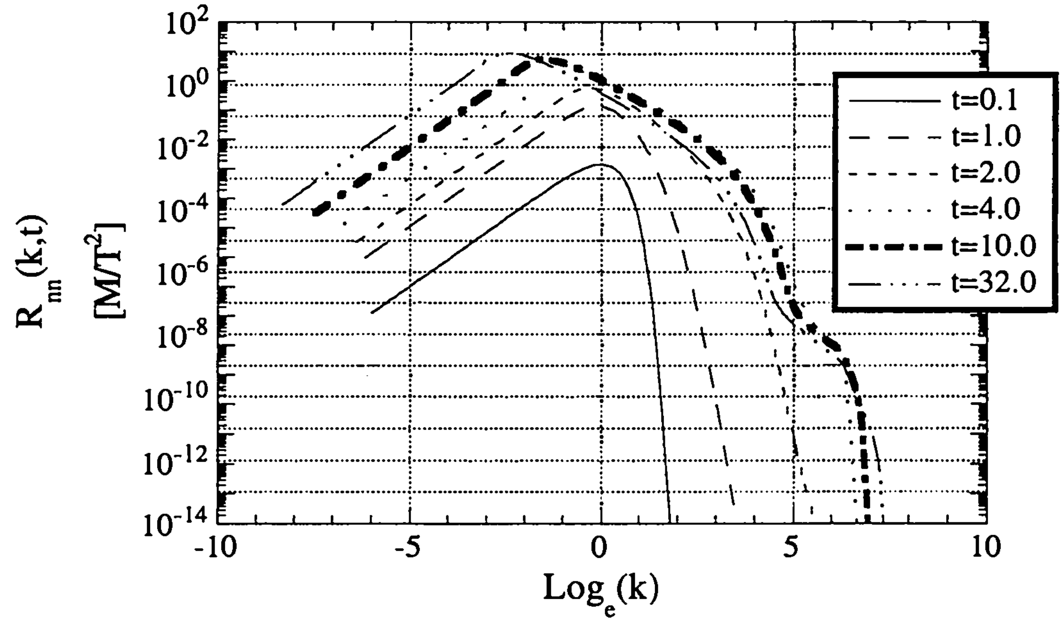

Figure 25. Evolution of the spectrum of $R_{nn}(k,t)$ during constant acceleration (Case 4).

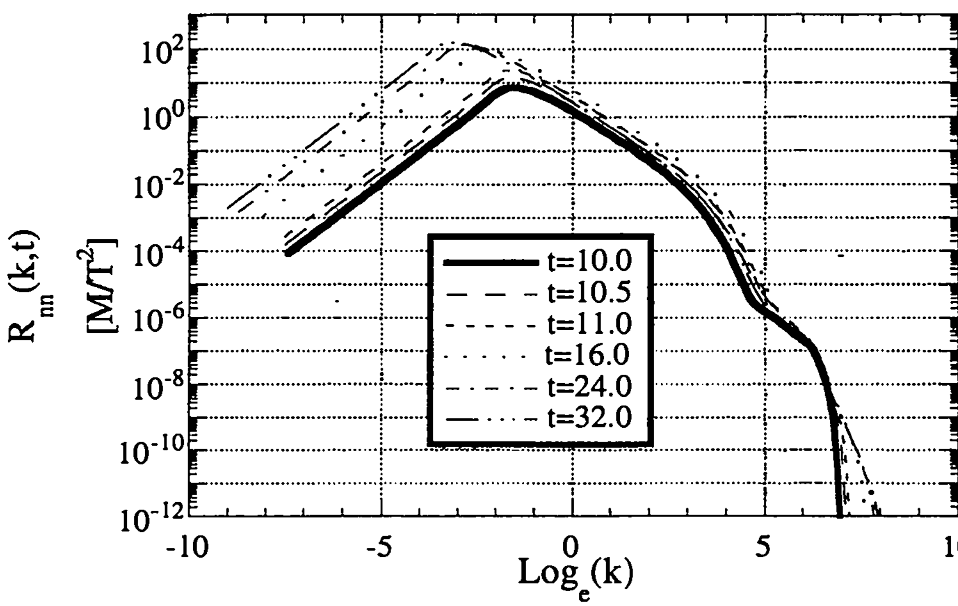

Figure 26. Evolution of the spectrum of $R_{nn}(k,t)$ after an increase in the acceleration (Case 5).

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