LA-UR -85-723

CONF-8407103--2

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

TITLE: ANALYSIS OF NONLINEAR PARABOLIC EQUATIONS MODELING PLASMA DIFFUSION ACROSS A MAGNETIC FIELD

LA-UR--85-723

DE85 007683

AUTHOR(S): James M. Hyman and Philip Rosenau

SUBMITTED TO: Proceedings of SIAM/AMS 1984 Conference on Nonlinear Systems of Partial Differential Equations in Applied Mathematics

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

ANALYSIS OF NONLINEAR PARABOLIC EQUATIONS MODELING
PLASMA DIFFUSION ACROSS A MAGNETIC FIELD

James M. Hyman and Philip Rosenau

ABSTRACT. We analyse the evolutionary behavior of the solution of a pair of coupled quasilinear parabolic equations modeling the diffusion of heat and mass of a magnetically confined plasma. The solution's behavior, due to the nonlinear diffusion coefficients, exhibits many new phenomena. In a short time, the solution converges into a highly organized symmetric pattern that is almost completely independent of initial data. The asymptotic dynamics then become very simple and take place in a finite dimensional space. These conclusions are backed by extensive numerical experimentation.

1. INTRODUCTION

We study the asymptotic behavior of a plasma slowly diffusing across a strong magnetic field.[5-8,12,17] In the initial value problem, the plasma has compact support and diffuses into the surrounding vacuum. In the initial boundary value problem, the plasma is confined within a finite domain and convective boundary conditions are imposed. Both models are mathematical idealizations of a more complex physical situation, nevertheless they provide theoretical insight to the dynamics of a plasma heat and mass diffusion.

In past studies, the decoupled problems for the diffusion of particles in an (essentially) isothermal plasma[3-4] and the diffusion of heat in a stationary plasma[1,13,18] have been analyzed. The coupling of these two processes is the source of many new phenomena that are not present in a single diffusion equation.[15]

The equations of motion we will study are

$$\rho_t = (D_1\rho_x)_x \quad , \tag{1a}$$

$$P_t = (\rho D_2 T_x)_x + (TD_1\rho_x)_x \quad , \tag{1b}$$

where $D_i = d_{oi}\rho^{\alpha_i} T^{\beta_i}$, $i = 1,2$, P is the plasma pressure, ρ is the density, T is ionic temperature, assumed to be equal to that of the electrons, and $P = \rho T$. The initial data is specified over a bounded domain

$$\rho(x,0) = \rho_o(x) \quad , \; P(x,0) = P_o(x) \quad , \; x \in (-x_o,+x_o) \quad . \tag{2}$$

The divergence form of Eqs. (1) guarantee that no additional energy or mass is added (or subtracted) after the process is initialized. An alternative form of (1b) can be obtained for T,

$$\rho T_t = (\rho D_2 T_x)_x + (D_1 \rho_x) T_x \quad . \tag{1c}$$

This form reveals the convective nature of second term on the right hand side of (1b) in a diffusive disguise. The rapid convection of temperature down the density gradients is a dominate force in the asymptotic behavior of the solution.

2. INITIAL VALUE PROBLEM

We first construct a self-similar solution of Eqs. (1) for $x \in (-\infty,\infty)$ and

$$\rho_o(x) = M_o\delta(x) \quad , \; P_o(x) = E_o\delta(x) \quad . \tag{3}$$

where $\delta(x)$ is the Dirac delta function. The appropriate self-similar solution to (3) satisfies

$$P(x,t) = \rho(x,t)E_o/M_o \quad , \; \rho(x,t) = f(\zeta)/t^{1/(2+\alpha_1)} \tag{4}$$

where

$$\zeta = x/[(M_o^{\alpha_1} t)^{1/(\alpha_1+2)}]$$

and

$$f(\zeta) = [\alpha_1(\zeta_f^2 - \zeta^2)/(2\alpha_1 + 4)]^{1/\alpha_1}$$

if $\zeta \le \zeta_f$ and $f(\zeta) = 0$ otherwise.

Note that the position of the diffusing front ζ_f depends only on the total mass M_o of the system and α_1. It follows from (5) that the self-similar solution describes an isothermally diffusing plasma with $T = E_o/M_o$.

Out of the many group invariant solutions, the one presented has been selected because of its key role in the late-time evolution of solutions with more complicated initial data. That is, if the self-similar solution shares the same mass M_o and energy E_o as another initial value problem,

$$M_o = \int_{-x_o}^{x_o} \rho_o(x)dx \quad , \; E_o = \int_{-x_o}^{x_o} P_o(x)dx \quad , \tag{5}$$

then irrespective of the initial distribution of the plasma, the self-similar solution is the leading term in the far-field description of the original problem. This behavior is a natural generalization of single equation case.

Since we have not yet obtained a rigorous proof of the attractive nature of the self-similar solution, we performed a series of numerical experiments to confirm this property.

The isothermal nature of the asymptotic solution dominates so strongly that the *specific form of the second diffusion coefficient is of no importance in this stage of the problem*. A typical rapid transition to the self-similar regime is shown in Figure 1. After the initial transients, the plasma is isothermal, Eq. (1b) merely duplicates Eq. (1a), and the solution dynamics is almost identical to the single diffusion equation case.

Fig 1. The initial transient solution of Eqs. (1) for $\alpha_1 = ½$, $\beta_1 = ½$, $\alpha_2 = 1$, $\beta_2 = 1$, $d_{01} = 1$, and $d_{02} = 5$. The initial conditions and solution are symmetric about the origin.

Previously,[14] it was shown that if a finite mass M_o is distributed over the whole space, then the thermal diffusion as given by

$$\rho_o(x)T_t = [A(T)]_{xx}$$

leads to the isothermalization of the medium if A satisfies $A(0) = 0$, $A'(0) \geq 0$ and $A'(T) > 0$ for $T > 0$. That is,

$$T(x,t) \to T_a \equiv \int_{-\infty}^{\infty} \rho_o(x)T(x,0)dx/M_o \quad ,$$

as might be anticipated on the basis of physical considerations. The diffusion of heat in a finite mass medium results in isothermalization of the medium, irrespective of how the mass is distributed.

3. INITIAL BOUNDARY VALUE PROBLEMS

3.1 The Reduced Problem.

In order to analyse the behavior of the solution of the initial boundary value problem, we first drop the convective term in (1c). Later this term will be restored and its impact evaluated. Thus for the time being, we consider the evolution of

$$\rho_t = [D_1(\rho,T)\rho_x]_x \quad ; \qquad D_1 = d_{01}\,\rho^{\alpha_1}\,T^{\beta_1} \quad ; \qquad (6a)$$

$$\rho T_t = [\rho D_2(\rho,T)T_x]_x \quad ; \qquad D_2 = d_{02}\,\rho^{\alpha_2}\,T^{\beta_2} \quad , \qquad (6b)$$

and $x \in [-1,1]$. We prescribe initial data for density and temperature together with homogenous convective boundary conditions

$$\rho_x \pm h_1\rho = 0 \quad , \quad T_x \pm h_2T = 0 \quad \text{at } x = \pm 1 \quad . \qquad (7)$$

These convective boundary conditions are physically more relevant and mathematically more tractable than Dirchlet boundary conditions ($h = \infty$).

As in the Cauchy problem, the asymptotic shape of the solution of Eqs. (6) is simple and can be easily classified. The solution evolves very quickly toward a universal diffusion mode which is almost independent of initial data. For this case, however, the highly organized diffusion pattern is mathematically represented by a time-space separable solution. Similar separable solutions are known before to play a key role in the evolution of the solution to a single nonlinear diffusion equation[1,3,4,14].

The analysis of these separable solutions is the central theme of this section. While such solutions are special cases because they must satisfy special initial data, they attract all initial data and hence play the key role in the asymptotic stage of problems with arbitrary initial data.

Although rigorously we can prove this proposition only for a subclass of the considered problem, extensive numerical experimentation has been used to give strong credence to them being global attractors.

Inserting the separable forms

$$\rho(x,t) = \phi_1(t)N(x) \quad , \; T(x,t) = \phi_2(t)\psi(x) \quad , \tag{8}$$

into Eqs. (6a) and (6b) leads to the following conditions

$$\dot{\phi}_1 = -\lambda_1\phi_1^{\alpha_1+1}\phi_2^{\beta_1} \quad ; \qquad \lambda_1 \geq 0 \tag{9a}$$

$$\dot{\phi}_2 = -\lambda_2\phi_1^{\alpha_2}\phi_2^{\beta_2+1} \quad ; \qquad \lambda_2 \geq 0 \tag{9b}$$

$$d_{01}\frac{d}{dx}N^{\alpha_1}\psi^{\beta_1}\frac{dN}{dx} + \lambda_1 N = 0 \quad ; \tag{10a}$$

$$d_{02}\frac{d}{dx}N^{\alpha_2+1}\psi^{\beta_2}\frac{d\psi}{dx} + \lambda_2 N\psi = 0 \quad ; \tag{10b}$$

and the spatial part of Eq. (7).

The relevant cases of the first integrals of motion for Eqs. (9) are

I. $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$,

$$\frac{\lambda_1}{\beta_2 - \beta_1}\phi_2^{\beta_1-\beta_2} + \frac{\lambda_2}{\alpha_2 - \alpha_1}\phi_1^{\alpha_2-\alpha_1} = C_0 \quad ; \tag{11a}$$

II. $\alpha \equiv \alpha_1 = \alpha_2$, $\beta \equiv \beta_1 = \beta_2$,

$$\phi_1 = (\phi_2/T_0)^{\lambda_1/\lambda_2} \quad ; \tag{11b}$$

where C_0, ρ_0, and T_0 are constants.

Even though Case II is degenerate, it is of considerable practical interest in many applications where D_1/D_2 is assumed to be constant (such as for the diffusion of a fully collisional plasma across a magnetic field).

Integration of Eq. (9) yields

$$\phi_1(t) = [T_0^{\beta}(t_0 + \Omega t)]^{-\lambda_1/\Omega} \quad , \; \Omega = \lambda_1\alpha + \lambda_2\beta \quad , \tag{12}$$

where t_0 is a constant. According to whether Ω is positive, zero, or negative, we refer to the solution ϕ_1 as decaying slowly (algebraic decay), exponentially, or fast (ϕ_1 vanishes in a finite time).

The time dependences of the solutions in Case I is given implicitly as

$$\phi_1 = (\zeta_0 + \lambda_1\alpha_1\zeta)^{-1/\alpha_1}, \quad \phi_2 = (\tau_0 + \lambda_2\beta_2\tau)^{-1/\beta_2}, \tag{13a}$$

where ζ_0 and τ_0 are constants of integration and

$$dt = d\zeta/\phi_2^{\beta_1} = d\tau/\phi_1^{\alpha_2} \tag{13b}$$

defines ζ and τ, the stretched time coordinates. Of course, ζ_o and τ_o are not independent, since they are related by Eq. (11a).

Unless either β_1 or α_2 vanishes, τ may be found only after the integration of Eqs. (13b) and (11a). Though the resulting Euler type integrals can be solved only implicitly, ϕ_1 and ϕ_2 can be evaluated asymptotically to determine the large time behavior. The results of this analysis are summarized in Fig. 2.

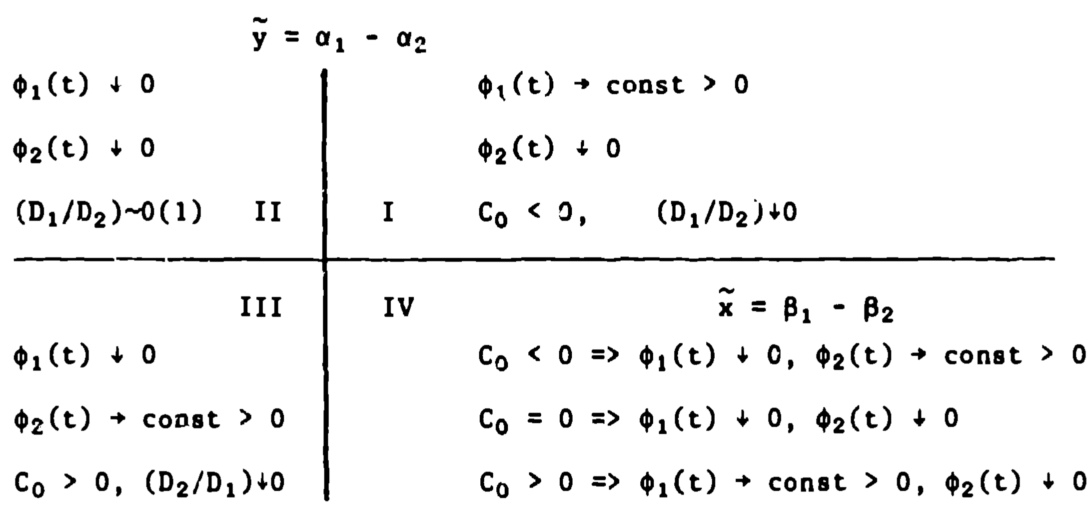


Fig. 2. Solution states of Eqs. (6) in the $(\tilde{x},\tilde{y}) = (\beta_1-\beta_2,\alpha_1-\alpha_2)$ plane. In the first and the third quadrant, the integration constant C_o must have a definite sign, but its value is irrelavant for solutions in the second quadrant, and crucial in the fourth quadrant. Everywhere, but on the $\Delta \equiv \alpha_2\beta_1 - \alpha_1\beta_2 = 0$ line, the decay is algebraic.

We can find important features of the solution's temporal part directly from the first integrals of motion. In the $(\tilde{x}, \tilde{y}) = (\beta_1-\beta_2,\alpha_1-\alpha_2)$ plane in Fig. 2, the two possible lines where $\Delta \equiv \alpha_1\tilde{x} - \beta_1\tilde{y} = 0$ separate regimes of fast and slow diffusion (the quantifier "fast" means that the process is extinguished within a finite time). The behavior of the temporal part of the solution dramatically changes in each of the four quadrants. In general, only in the second quadrant do both $\phi_1(t)$ and $\phi_2(t)$ decay to zero, elsewhere one of the ϕ's converges to a positive constant (see 11a).

For large t, the asymptotic form of $\phi_i(t)$ in the second quadrant is given by

$$\phi_i(t) = (t_o+\lambda_i\omega_i t)^{\omega_i}, \; i = 1,2 \quad , \tag{14a}$$

where

$$\omega_1 = (\beta_2-\beta_1)/\Delta \quad , \quad \omega_2 = (\alpha_1-\alpha_2)/\Delta \quad ; \quad \Delta \equiv \alpha_2\beta_1 - \alpha_1\beta_2 \quad . \tag{14b}$$

The decay to zero is algebraic as described by Eq. (14), everywhere but on the lines where Δ is zero, the solution decay is exponential.

The ω_1 and ω_2 which give the temporal decay rates are defined a priori, and are independent of the symmetry in which our problem is considered. This is an essential feature of the nonlinear diffusion which has no counterpart in the linear theory.

To obtain the main features of the temporal behavior in the other quadrants one can use Eqs. (9) along with the fact that the first integral of motion (11a) forces one of the ϕ_i's to approach a non-zero constant everywhere but in the second quadrant. That is, first assume that

$$\phi_1 \cong \phi_{10} = \text{const.} > 0 \tag{15a}$$

then from (9b) we have

$$\phi_2 \cong (t_0 + \delta_A t)^{-1/\beta_2} \quad , \; \delta_A \equiv \lambda_2\beta_2\phi_{10}^{\alpha_2} \quad . \tag{15b}$$

Inserting (15b) into (9a) we get a correction to ϕ_{10} and a consistency relation $\beta_1 > \beta_2$ for (15a) to hold.

Proceeding in a similar fashion with ϕ_2, we assume

$$\phi_2 \cong \phi_{20} = \text{const.} > 0 \tag{16}$$

then from (9a) we have

$$\phi_1 \cong (t_0 + \delta_B t)^{-1/\alpha_1} \quad , \quad \delta_B \equiv \lambda_1 \alpha_1 \phi_{20}^{\beta_2} \quad , \tag{17a}$$

which in turn, when inserted into (9b) yields

$$\phi_2 \cong [a_1 - a_2 (t_0 + \delta_B t)^{-d_1}]^{-1/\beta_2} \quad , \tag{17b}$$

$$d_1 = (\alpha_2 - \alpha_1)/\alpha_2 \quad , \quad a_i = \text{const.} > 0 \quad ,$$

and a consistency relation, $\alpha_2 > \alpha_1$. In the fourth quadrant either ϕ_1 or ϕ_2 may tend to a constant.

The rate of the temporal decay, is intimely related to the role played by the separation constants λ_1 and λ_2. To clarify this point consider first the case when Eq. (6) is a linear system whose solution decays as $\exp(-\lambda_i t)$, where λ_1 and λ_2 play the role of eigenvalues in Eqs. (10). In a nonlinear diffusive system, the λ_i are nonessential constants in Eqs. (10) whose values depend on the normalization of ψ and N. Indeed, suppose that $\psi(0) = A$ and $N(0) = B$ with $\tilde{\psi}$ and $\tilde{N}$ being the solutions with eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. For any ψ_0, $N_0 > 0$, we then find that $\psi = \psi_0\tilde{\psi}$ and $N = N_0\tilde{N}$ are also solutions with $\tilde{\lambda}_i \rightarrow \tilde{\lambda}_i N_0^{\alpha_i} \psi_0^{\beta_i}$, $i = 1,2$. Alternatively, let $\Delta = \alpha_2\beta_1 - \alpha_1\beta_2$, then choosing

$$\psi_0^{\Delta} = \tilde{\lambda}_2^{\alpha_1}/\tilde{\lambda}_1^{\alpha_2} \quad , \quad N_0^{\Delta} = \tilde{\lambda}_1^{\beta_2}/\tilde{\lambda}_2^{\beta_1} \quad , \tag{18}$$

normalizes both λ_1 and λ_2 to one, with $\psi(0) = A\tilde{\Psi}_0$ and $N(0) = B\tilde{N}_0$.

Thus the λ's may be reshuffled from the spatial into the temporary part of the solution and are related to the amplitude of the diffusion mode (e.g., see Eqs. (14)). This relationship is fundamentally different from the linear case.

An exception occurs when Δ vanishes. The linear case is a trivial example. In the nonbanal case, where $\alpha_1/\alpha_2 = \beta_1/\beta_2 \neq 0$ (or ∞), only one λ can be eliminated from Eqs. (10); the other λ remains as an essential parameter. In this case, the solutions to Eqs. (6) are invariant with respect to the group of shifts; $T \rightarrow AT$, $\rho \rightarrow A^{-\beta_1}/\rho^{\alpha_1}$, and $t \rightarrow t + t_0$. If $A = \exp(-\lambda t_0)$, this

this invariance allows solutions of the form

$$T = e^{\lambda t}\psi(x) \quad , \quad \rho = e^{-\beta_1 \lambda t/\alpha_1} N(x) \quad , \tag{19}$$

where λ is an eigenvalue that must be determined from the global existence conditions of the separable solution. (A similar situation arises in the problem of imploding shock waves, where the λ is determined uniquely by requiring the existence of the self-similar solution in the large.[2,18])

A physically interesting case arises when D_1/D_2 is constant and (Case II, Eq. (11b)). Again (λ_1/λ_2) plays the role of an eigenvalue with the exponential case being a transit solution between fast and slowly diffusing regimes. Here, both the mass and energy decay algebraically at a rate λ_i/Ω, i = 1, 2. (See Eqs. (11b) and (12)) that must be found by solving Eqs. (10a) and (10b).

For given convective boundary condition coefficients h_1 and h_2, the following homologous property:

$$\lambda_2 \, d_{01} \, /(\lambda_1 \, d_{02}) \; = K_0 \tag{20}$$

means that λ_1/λ_2 has to be only measured for one pair of d_{01} and d_{02} and then it may be calculated for any other d_{01} and d_{02}. Particularly, if $\alpha\beta < 0$, such as in the fully collisional plasma case wherein $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = -\frac{1}{2}$, by changing the ratio of d_{01}/d_{02} we may transit from fast into a slow diffusion regime (or vice versa).

Having delineated the temporal part of the solution, we still need to interpret the fact that in a diffusive process when α and β are not in the second quadrant, one of the solutions (i.e., either ϕ_1 or ϕ_2) does not decay to zero. The time evolution of a particular example is shown in Fig. 3. This behavior is very different from what is expected from a single diffusion equation.

To understand the principle mechanism involved in this somewhat unexpected process, consider the case where β_1 is zero, Eqs. (6a) and (6b) decouple and can be solved separately. The separable solution of Eq. (6a) is a global attractor[1,3,14] and represents a universal mode of diffusion with the temporal behavior

$$\phi_1(t) = (t_0 + \lambda_1\alpha_1 t)^{-1/\alpha_1} \quad , \quad t_0 = \text{const.} \quad ; \tag{21}$$

Figure 3. For this initial data (symmetric about the origin) and these parameters in the first quadrant, $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 1$, $\beta_2 = \frac{1}{2}$, $d_{01} = 1$, $d_{02} = 5$, $h = -10$ the decay and diffusion of mass under Eqs. (6) is inhibited by the rapid decay of heat.

and $\phi_2(t)$ is given by Eq. (11a). If α_1 is positive, the solution asymptotically converges to the separable form. In numerical tests, the general solution becomes indistinguishable from the separable one after a relative short time. The constant t_0 depends upon the initial data. For a single equation t_0 is important only in the case of fast diffusion when $t_0/(\lambda_1|\alpha|_1)$ defines the finite extinction time of the process.

Although $\phi_2(t)$ is known from Eq. (11a), analysing the solution of Eq. (6b) directly is instructive. Using the asymptotic form of ρ, known for Eq. (6a), we can treat Eq. (6b) as a separate equation in T with a variable diffusion coefficient. Numerically, we have found that the solution of this equation rapidly converges to this asymptotic separable form. With this expectation, we substitute $\rho = \phi_1(t)N(x)$ and obtain

$$N(x)\phi^{-\alpha_2} T_t = N(x)T_\tau = (N^{\alpha_2+1} T^{\beta_2} T_x)_x \quad , \tag{22}$$

where

$$\tau = \int_0^t \phi_1^{\alpha_2}(\eta)d\eta \quad . \tag{23}$$

When $0 < N < \infty$, Eq. (22) is a standard diffusion equation, similar to Eq. (6a) with $\beta_1 = 0$, but measured in τ units.

If $\beta_1 = 0$ and $\alpha_1 \geq \alpha_2$, then $\tau \to \infty$ as $t \to \infty$. For large τ-time, temperature converges to the separable solution $T = \phi_2(\tau)\ \psi\ (x)$ with

$$\phi_2(t) = \tilde{\phi}_2[(\tau(t)] = (\tau_0 + \lambda_2\beta_2\tau)^{-1/\beta_2} \quad , \tag{24}$$

and again τ_0 is an unknown function of the initial conditions.

If $\beta_1 = 0$ and $\alpha_2 > \alpha_1$ the integral in Eq. (23) converges, and

$$\tau = \tau_D[1 - (1 + \lambda_1\alpha_1 t/t_0)^{1-\alpha_2/\alpha_1}] \quad , \tag{25a}$$

where

$$\tau_D = t_0^{1-\alpha_2/\alpha_1}/[\lambda_1(\alpha_2 - \alpha_1)] \quad . \tag{25b}$$

Thus, $\tau \to \tau_D$ as $t \to \infty$. If τ is bounded, the time needed to attain the separable solution is not available, and $\phi_2(t \to \infty)$ converges to a positive constant. Thus, while $\rho(x,t \to \infty)$ decays to zero, $T(x,t \to \infty) \to T(x,\tau_D)$ is a positive nonzero steady state.

When this is the case, the asymptotic temperature will remember its initial conditions. If, in addition, $\beta_1 = \beta_2 = 0$, then this follows at once by noting $T(x,t) = \sum^{\infty} a_j \cdot \exp[-\delta_j\tau(t)]\psi_j(x)$. Here δ_j and ψ_j are the j^{th} eigenvalue and eigenfunction, respectively. Using Eqs. (25), we can see from

$$T(x,\ t \to \infty) \to T(x,\ \tau_D) = \sum^{\infty} a_j\ \exp(-\delta_j\tau_p)\psi_j \tag{26}$$

that none of the harmonics initially present vanish as $t \to \infty$.

For the non-linear case we show this property by taking $\psi(x)$, the spatial counterpart of (22), as the initial condition and perturbing it. The perturbed solution of Eq. (22) is

$$T(x,\ t) = \phi(t)\psi(x)\ [1 + u(x,\ t)] \quad . \tag{27}$$

If $u = w(t)\ V(x)$, then ψ is the first eigenfunction of V. Again $w(\infty) = w(\tau_D) > 0$ and u cannot return to $\phi_2\psi$.

Thus, in the third quadrant where $\beta_2 > \beta_1$, $\alpha_2 > \alpha_1$, the diffusion of heat is always inhibited by the fast diffusion of density. In the fourth quadrant, depending on the initial data and the values of α_i and β_i, either temperature or density will inhibit the diffusion of the other.

Numerical experiments have shown that usually the density decays faster and inhibits the diffusion of heat, as in the third quadrant. If α_1 is negative, the process always terminates on the fast scale. If α_1 is positive, the process is ***fast*** if the ***temperature vanishes*** and the plasma becomes cold within a finite time, but it is ***slow*** if the ***density decays to zero***.

When $\beta_1 \neq 0$, the asymptotic analysis of the temporal part is more tedious but confirms the above conclusions. However, for $\beta_1 \neq 0$ we were unable to analytically demonstrate the attractive nature of the separable solution. It is at this point that an extensive numerical experimentation was used covering all of the four quadrants of the $(\tilde{x}, \dot{y})$ plane to ensure the attractive nature of the separable solution. This leads us to believe that the lack of rigorous mathematical proof is rather a technical than a fundamental obstacle. Moreoever if $\tau_D < \infty$, unlike the semicoupled case, either both T and ρ come close to their ideal counterparts ψ and N or neither comes close, as $\tau \to \tau_D$. In practice however, for the many cases considered numerically, T and ρ approach their attracting separable solutions very quickly, long before the process "runs out of time." That is, by the time the diffusion coefficient becomes suppressed, the process is extremely close to its universal mode.

Fig. 4.1a. Density, $\rho(0,0) = 10$.

Fig. 4.1b. Temperature, $T(0,0) = 1$.

Fig. 4.2a. Density, $\rho(0,0) = 1$.

Fig. 4.2b. Temperature, $T(0,0) = 20$.

Fig. 4. Symmetric solutions of the diagonal case, Eq. (6), with parameters in the fourth quadrant, $\alpha_1 = -\frac{1}{4}$, $\alpha_2 = -\frac{1}{2}$; $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{1}{4}$, $d_{01} = 1$, $d_{02} = 5$, $h = -10$, either the density or the temperature may decay to zero in a finite time, leaving the other stranded.

In Figs. 4, we show two examples with parameters in the fourth quadrant of how either temperature or density diffusion becomes depressed. The initial conditions and solution are shown for a massive relatively cold plasma where temperature vanishes in a finite time (Fig. 4.1), and a hot relatively tenuous plasma, where density decays to zero in a finite time (Fig. 4.2). In Fig. 4.2 the maximum initial temperature is T(0.0) = 20. If T(0,0,) = 10 then both components decay faster than exponentially and race toward zero between ρ and T ends as it does in 4.2 but with the final temperature several orders of magnitude smaller.

3.2 The Tensorial Case.

We are now in the position to discuss initial boundary value problems for the tensorial system Eqs. (1). The evolution of the temperature and the effect of the convective term in (1c) is more easily understood by working with this equation rather than (1b).

Substituting yields the separable form (8) into (1c) yields

$$d_{02} \frac{d}{dx} (N^{1+\alpha_2} \psi^{\beta_2} \frac{d\psi}{dx}) + d_{01} S(t) (N^{\alpha_1} \psi^{\beta_1} \frac{dN}{dx}) \frac{d\psi}{dx} + \lambda_2 N \psi = 0 \tag{28}$$

where

$$S(t) = \phi_1^{\alpha_1-\alpha_2} \phi_2^{\beta_1-\beta_2} .$$

Compare this equation with (10b). The status of (28) depends critically on the behavior of S(t). In turn, the behavior of S(t) critically depends on which quadrant of the $(\tilde{x},\tilde{y})$ plane the parameters reside. The possible behaviors are:

1st quadrant: $S(t) \downarrow 0$

2nd quadrant: $S(t) = 0(1)$

3rd quadrant: $S(t) \to \infty$

4th quadrant: if $\begin{cases} \phi_1 \to \text{const.}, & S_1(t) \downarrow 0 \\ \phi_2 \to \text{const.}, & S_1(t) \to \infty \end{cases}$.

In the first quadrant, asymptotically the convective term becomes completely suppressed and the shape of both N and ψ remain unaffected by the convective part. In the s . quadrant, S(t) is a constant which modifies the shape of the eigenfunctions ψ and N. Otherwise the characterization of the solution in this quadrant does not change.

S(t) has the most dramatic impact in the third quadrant. Here, S(t) will grow indefinitely unless the temperature becomes isothermal. But, the boundary conditions for T in Eqs. (7) prevent this if $h_2 \neq 0$. Asymptotically, this difficulty is resolved by T converging to a constant everywhere but near the boundary, where an ever thinning boundary layer will be present. A numerical example of such a case is shown in Fig. 5 and should be compared to the diagonal tensor case in Fig. 4. Since the temperature is nearly constant everywhere except for a small boundary layer, the dynamics of the problem are confined primarily to the density Eq. (1a).

Fig. 5.1a. Density, $\rho(0,0) = 10$.

Fig. 5.1b. Temperature, $T(0,0) = 1$.

Fig. 5.2a. Density, $\rho(0,0) = 1$.

Fig. 5.2b. Temperature, $T(0,0) = 10$.

Fig. 5. Symmetric solutions of tensorial case, Eqs. (1), with parameters in the fourth quadrant, $\alpha_1 = -\frac{1}{4}$, $\alpha_2 = -\frac{1}{2}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{1}{4}$, $d_{01} = 1$, $d_{02} = 5$, $h = -10$ either the density or the temperature may decay to zero in a finite time, leaving the other stranded.

In the fourth quadrant the situation is, as in Eqs. (6), either an extension of the first or of the third quadrant.

Finally, note that if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, $S(t) = O(1)$ and, as in the diagonal case, the decay rate is unknown a priori and the selected pattern depends upon the initial data.

4. NUMERICAL CALCULATIONS

Several hundred numerical experiments were performed to support the claims made about the stability and self-similarity of the asymptotic solutions. In the calculations, we used second-, fourth- and sixth-order centered finite difference approximations[11] on grids ranging from 20 to 200 mesh points on a CRAY X-MP computer. The boundary conditions were incorporated by extrapolating the solution to fictitious points outside the region where the solution was being integrated.[9] The cubic extrapolant satisfied both the boundary conditions and the differential equation at the boundary. The solution was integrated in time using a variable order - variable time step method of lines code, MOL1D,[10] that retained an absolute error tolerance between 10^{-4} to 10^{-6} per unit time step. Many problems were recalculated several times with different order finite difference approximations in space, grid resolution and time truncation error criteria to insure the numerical solutions were converged within an acceptable accuracy.

5. SUMMARY

The dynamics of the highly coupled quasilinear equations (1), is surprisingly simple. After a relatively short transit time the dynamics takes place in a finite dimensional space and is almost independent of the choice of initial data. In the initial value problem the medium quickly becomes isothermal and the dynamics are confined to mass diffusion. The initial boundary value problem offers a much wider variety of phenomena, all of which depends on the choice of the nonlinear diffusion coefficients D_1 and D_2. Among the phenomena which do not have a counterpart in the single diffusion equation case are:

1) The diffusion is at an unknown a priori rate and the density and temperature are similarity solutions of the second kind.
2) The diffusion rate of one solution component vanishes in favor of the other. The faster decaying solution component is predetermined in quadrants II and III of the $(\beta_1-\beta_2, \alpha_1-\alpha_2)$ plane.
3) In quadrant IV the decay is reminiscent of pattern selection where the winning solution component depends upon the initial data; which in turn decides which diffusion pattern is chosen.

Although we have extensive numerical calculations, the mathematical status of the problem is that we know everything (almost) about the evolution of Eqs. (1) but can prove nothing (almost).

Acknowledgements

The first author wishes to express his gratitude to the Center for Nonlinear Studies and the CTR Division for sponsoring his visit to the Los Alamos National Laboratory.

References

1. D. G. Aronson and L. A. Peletier, J. Diff. Eqs. 39, 378 (1981).

2. G. I. Barenblatt, Similarity, Self-similarity, and Intermediate Asymptotics, New York, Consultants Bureau, 1979.

3. J. Berryman, J. Math. Phys. 18, 2108 (1977).

4. J. Berryman and C. J. Holland, Phys. Rev. Lett. 40, 1720 (1978).

5. D. F. Düchs, D. E. Post, and P. H. Rutherford, Nucl. Fusion 17, 565 (1977).

6. H. Grad, Courant Inst. Report MF-95/1979.

7. J. T Hogan, "Multifluid Tokamak Transport Models," Methods in Computational Physics (Academic, New York, 1976), Vol. 16, 131.

8. W. A. Houlberg and R. W. Conn, Nucl. Fusion 19, 81 (1979).

9. J. M. Hyman, "Numerical Methods for Nonlinear Differential Equations," Nonlinear Problems: Present and Future, A. R. Bishop, D. K. Campbell, B. Nicolaenko, eds., North-Holland Publishing Co., (1982), 91-107.

10. J. M. Hyman, "The Method of Lines Solution of Partial Differential Equations," Courant Institute of Mathematical Sciences, Vol. COO-3077-139 (October 1976).

11. J. M. Hyman and B. Larrouturou, "The Numerical Differentiation of Discrete Functions Using Polynomial Interpolation Methods," Appl. Math and Comp., Vols. 10-11; also published in Numerical Grid Generation, J. F. Thompson, Ed., Elsevier North Holland, New York (1982), 487-506.

12. E. F. Jaeger, and C. L. Hedrick, Jr., Nucl. Fusion 19, 443 (1979).

13. S. Kamin and P. Rosenau, Comm. Pure Appl. Math., 34, 831 (1981).

14. S. Kamin and P. Rosenau, J. Math. Phys. 23, 1385 (1982).

15. P. Rosenau and J. M. Hyman, "An Analysis of Nonlinear Mass and Energy Diffusion," (1984) submitted to Physical Review A.

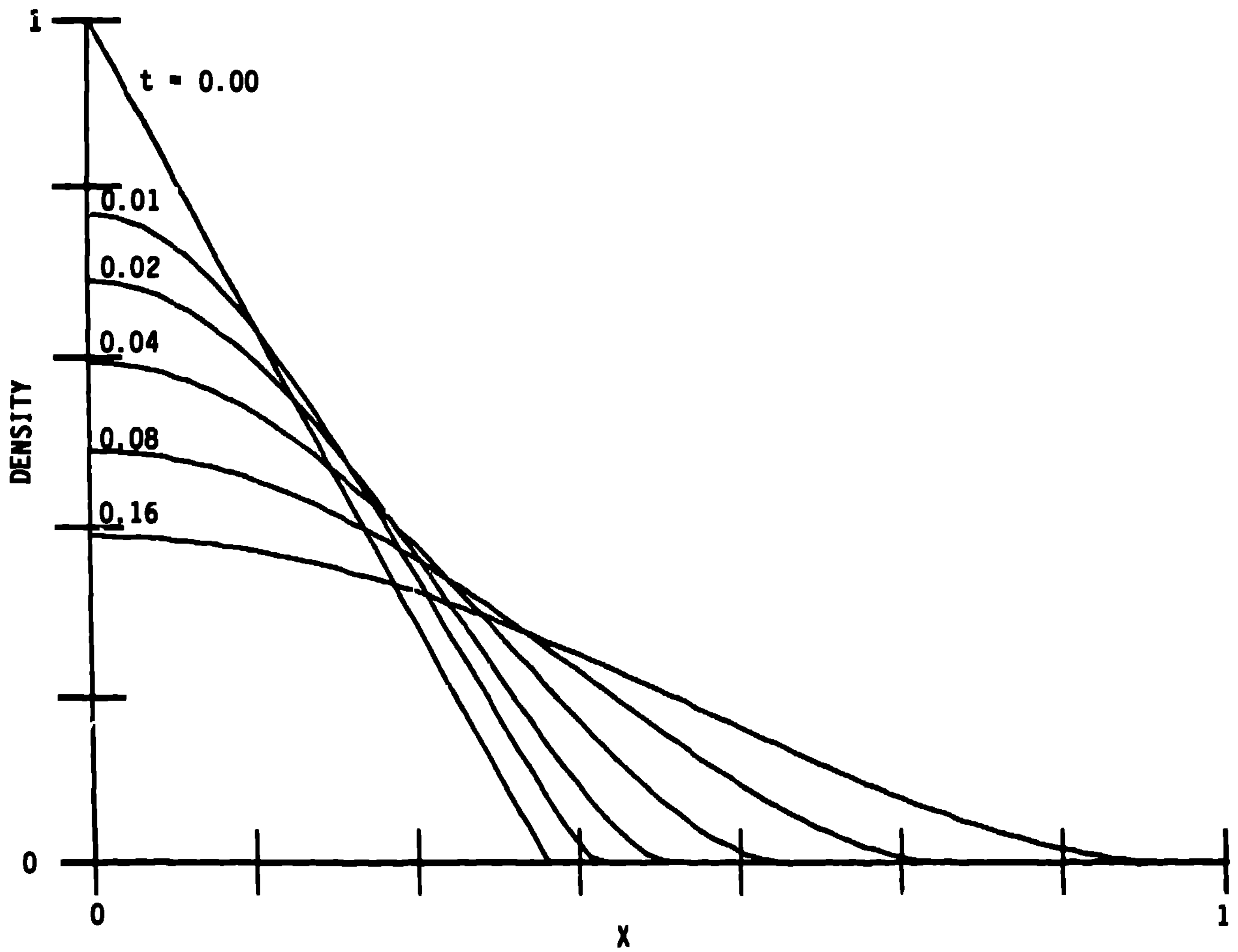
16. P. Rosenau and S. Kamin, Comm. Pure Appl. Math., 35, 113 (1982).

17. M. N. Rosenbluth and A. N. Kaufman, Phys. Rev. 109, 1 (1958).

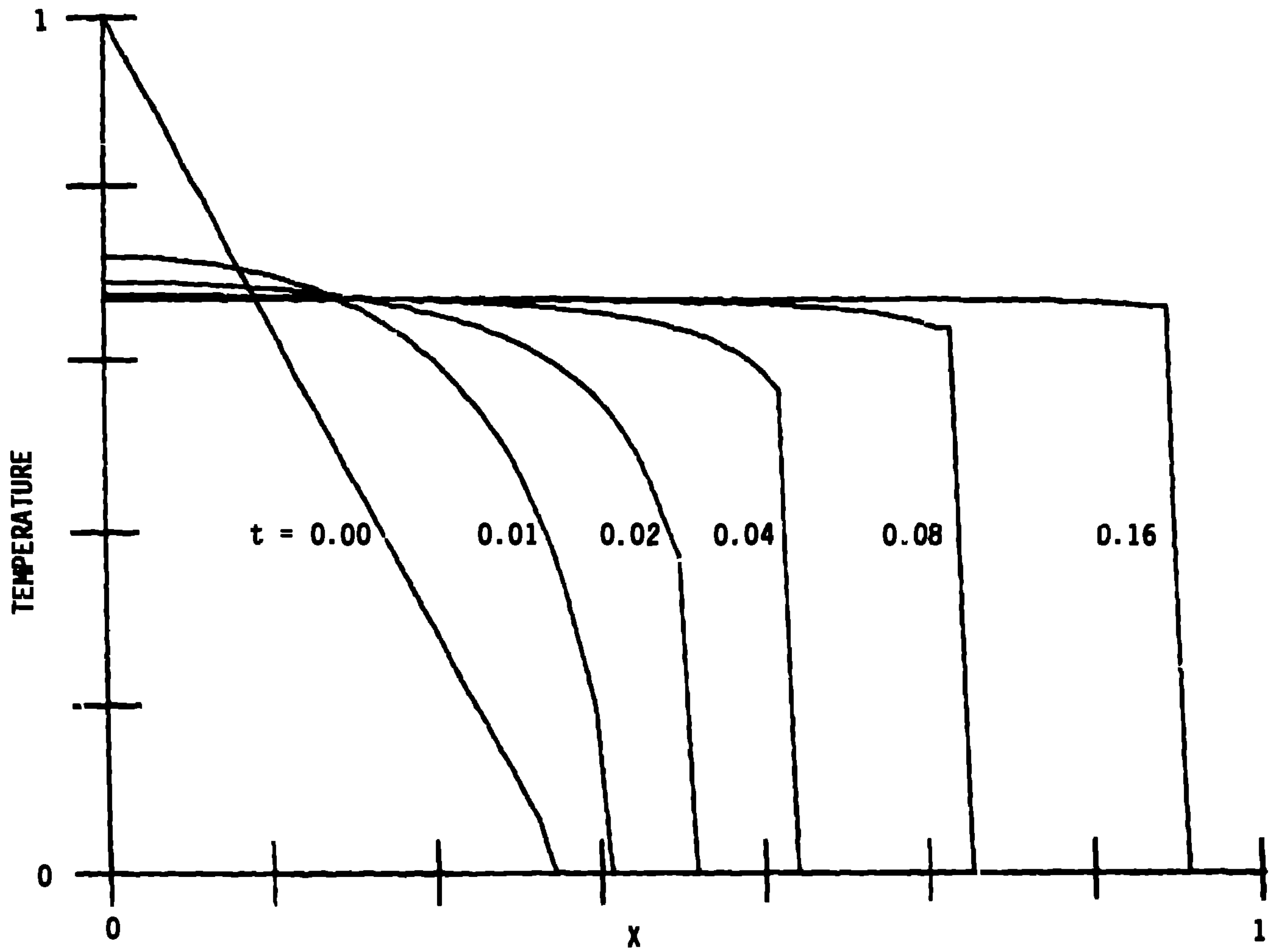
18. Y. A. B. Zeldovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, (Academic, New York, 1966).

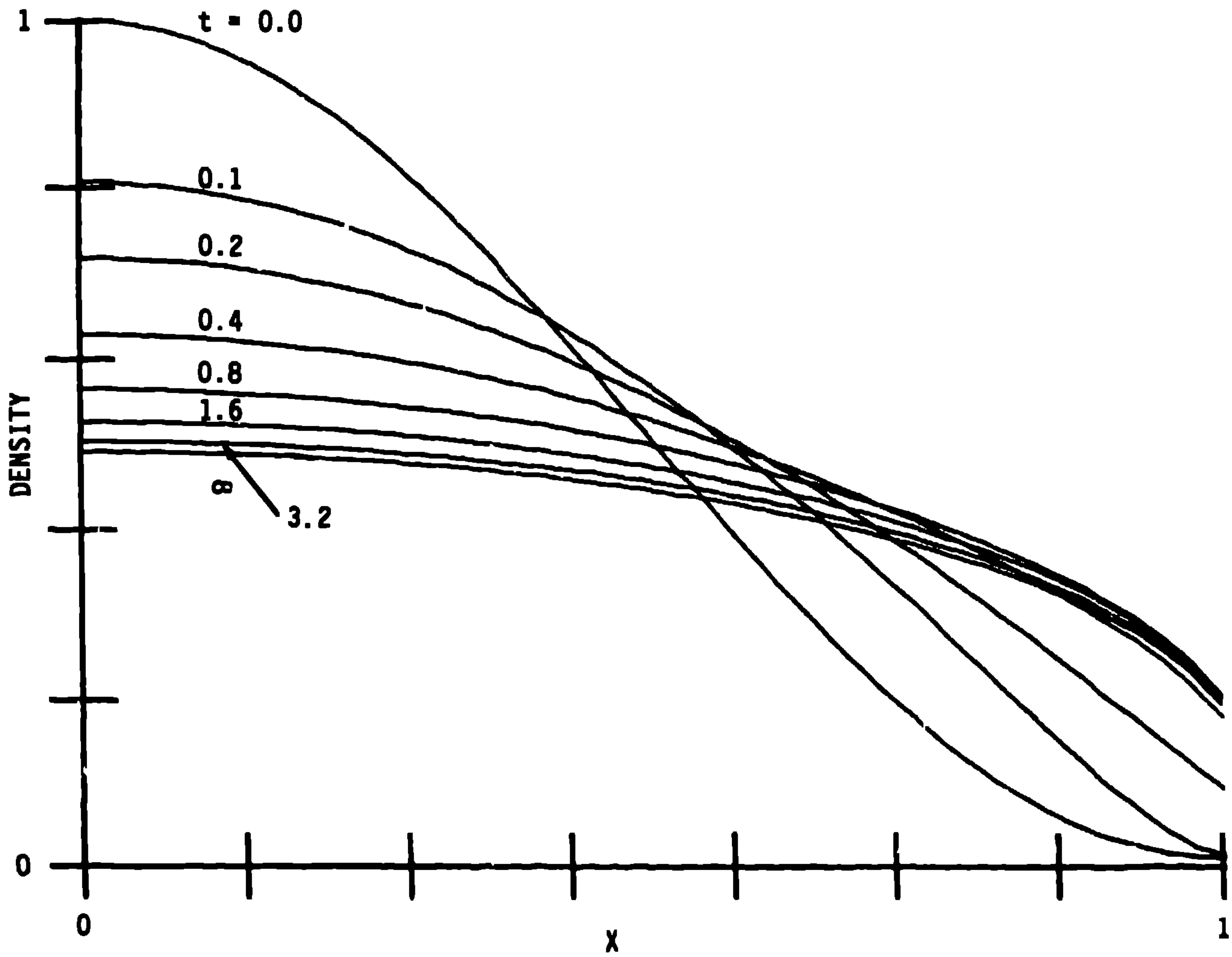
James M. Hyman
Center for Nonlinear Studies
Theoretical Division, MS B284
LOS ALAMOS NATIONAL LABORATORY
Los Alamos, NM 87545

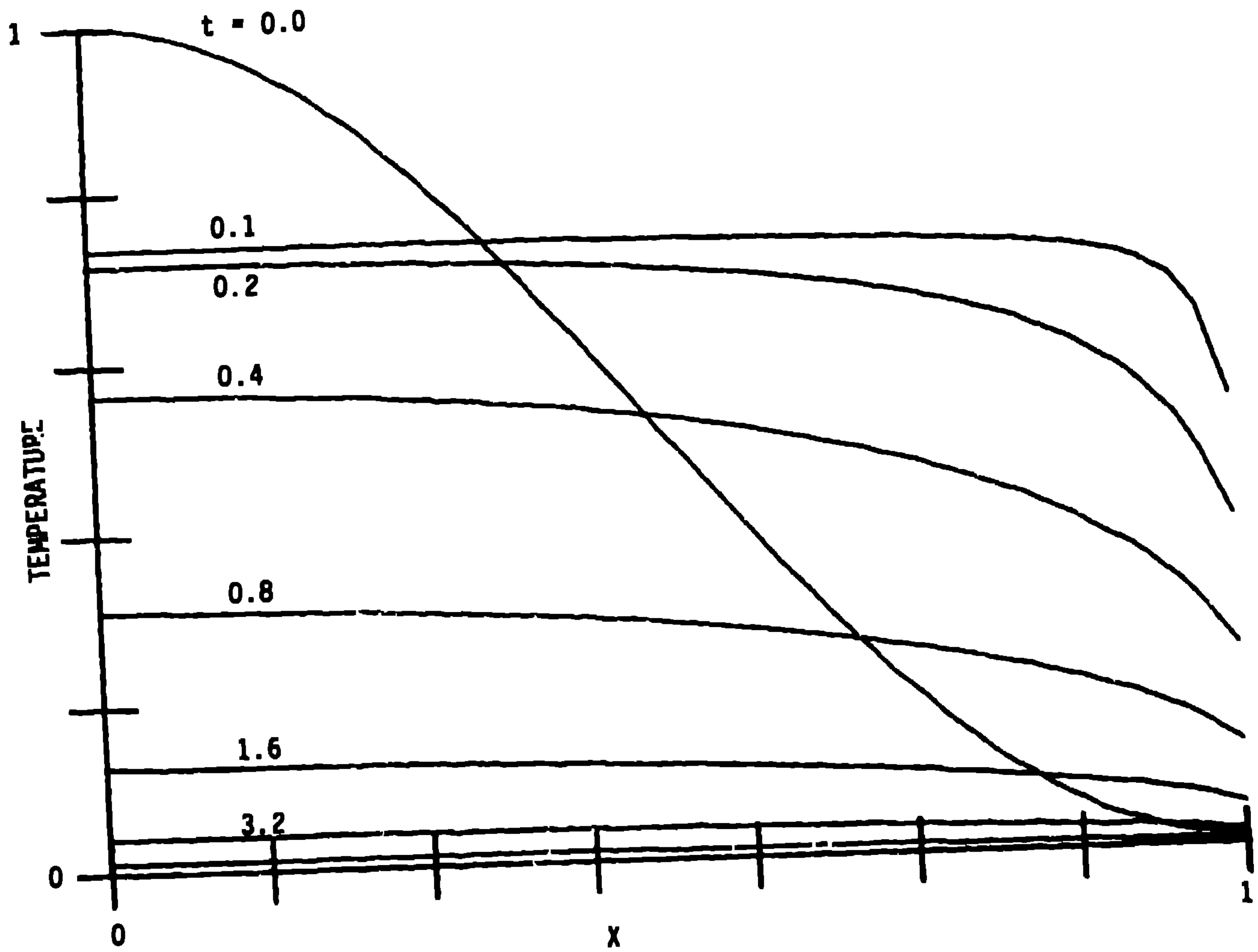
Philip Rosenau
Department of Mechanical Engineering
TECHNION
Haifa 3200
ISRAEL



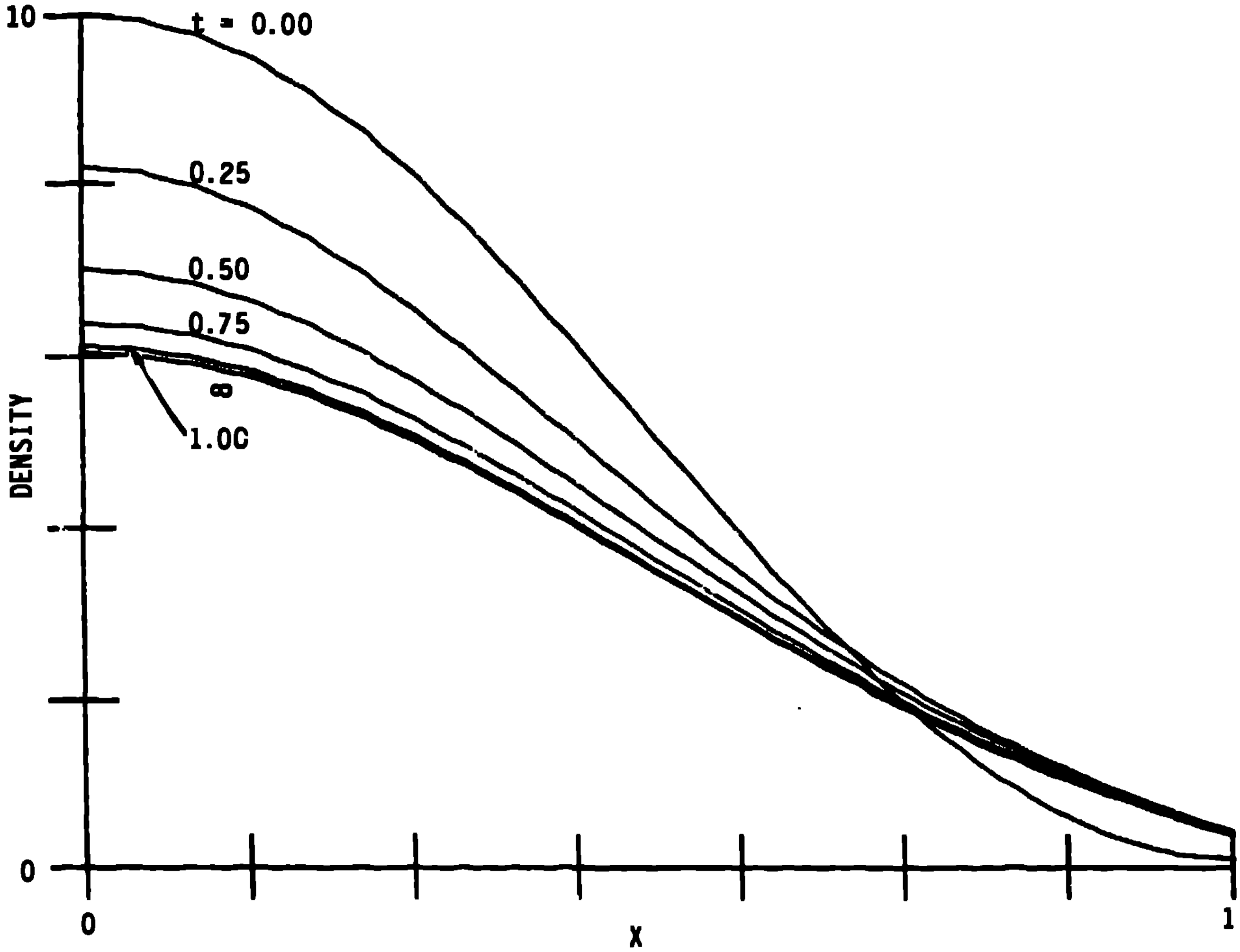
1a


1
0
TEMPERATURE
t = 0.00
0.01
0.02
0.04
0.08
0.16
0
X
1

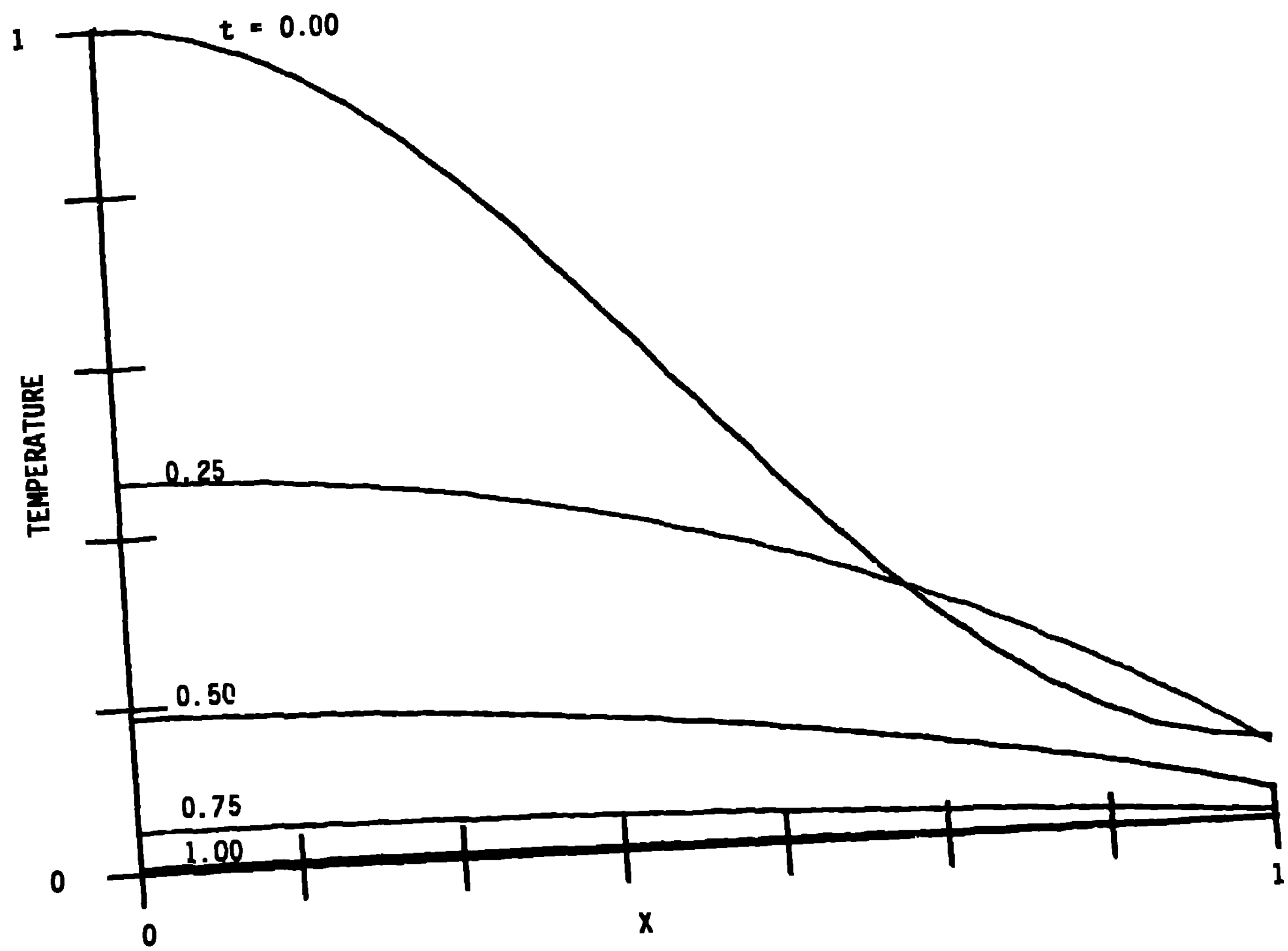

t = 0.0
0.1
0.2
0.4
0.8
1.6
∞
3.2
1
0
DENSITY
0
X
1



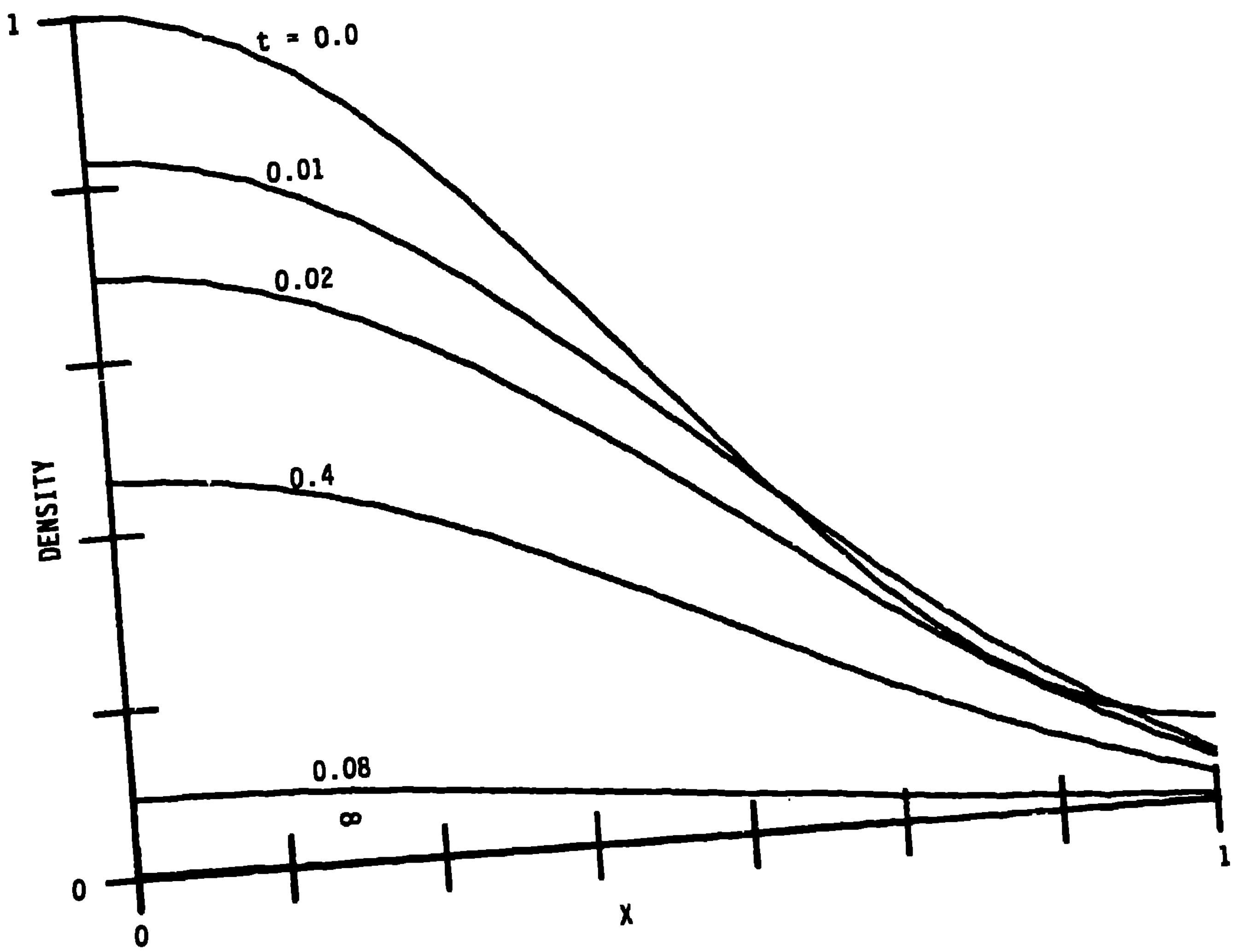
t = 0.0
0.1
0.2
0.4
0.8
1.6
3.2
1
0
TEMPERATURE
0
X
1



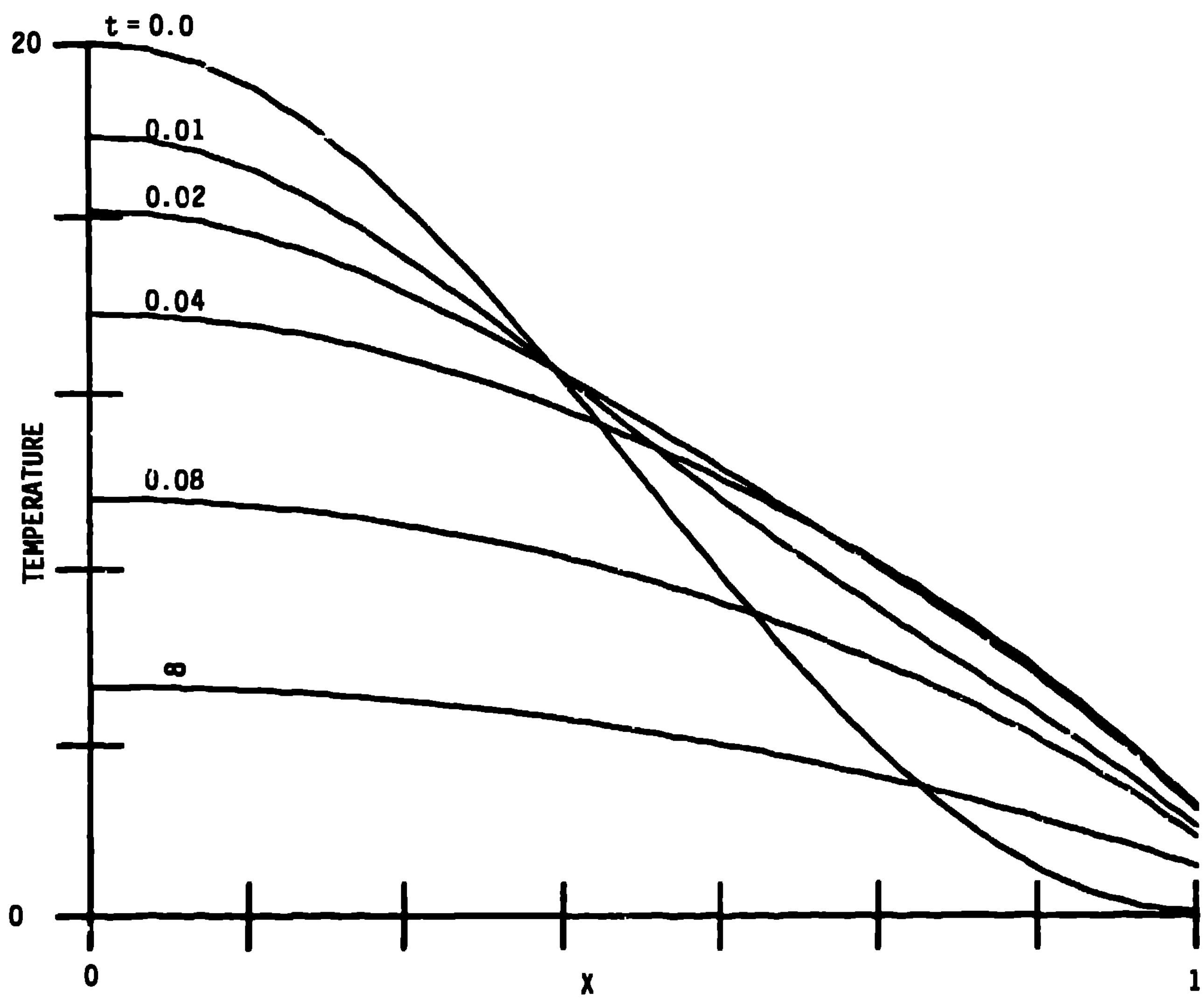
4.1a

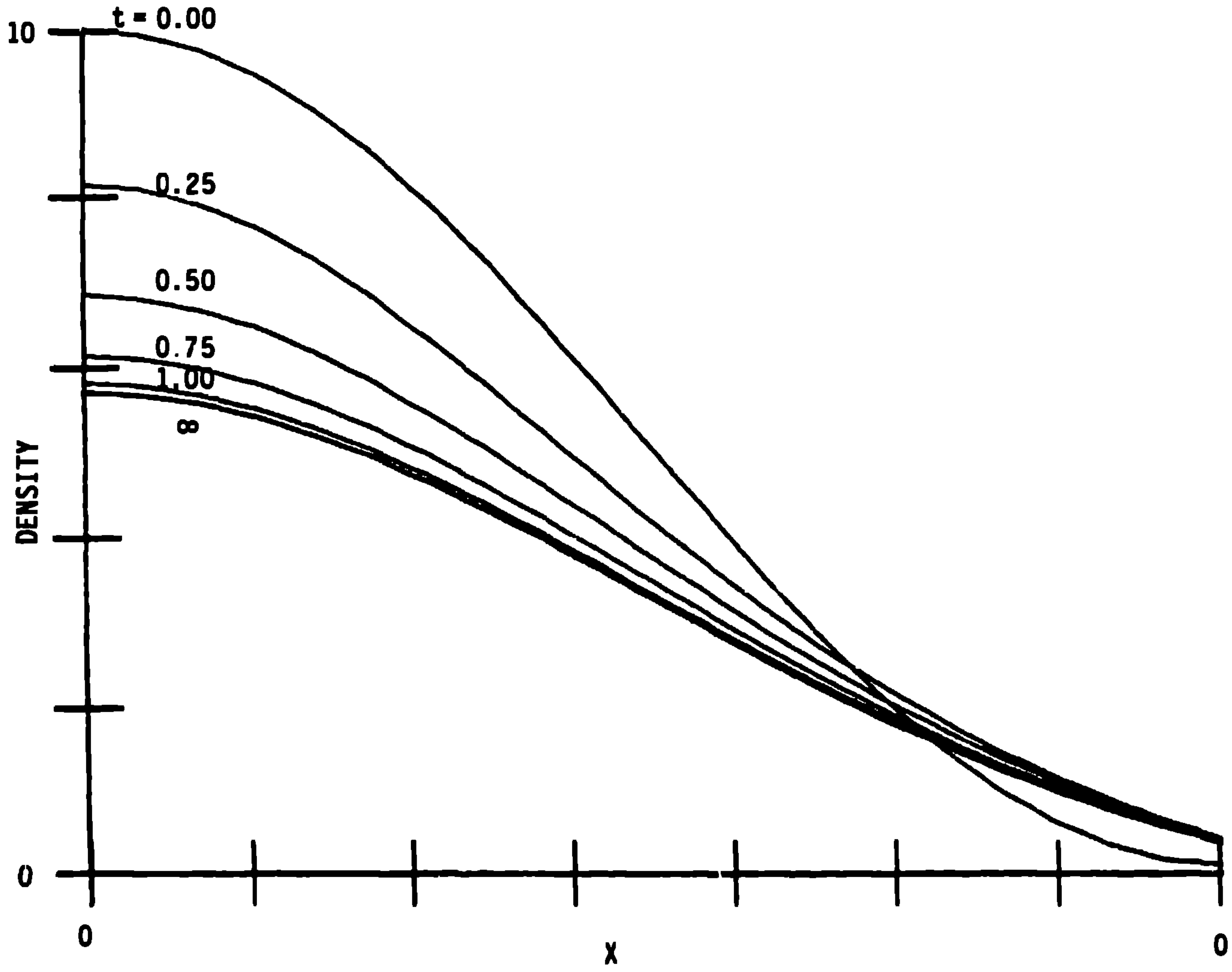


4.16

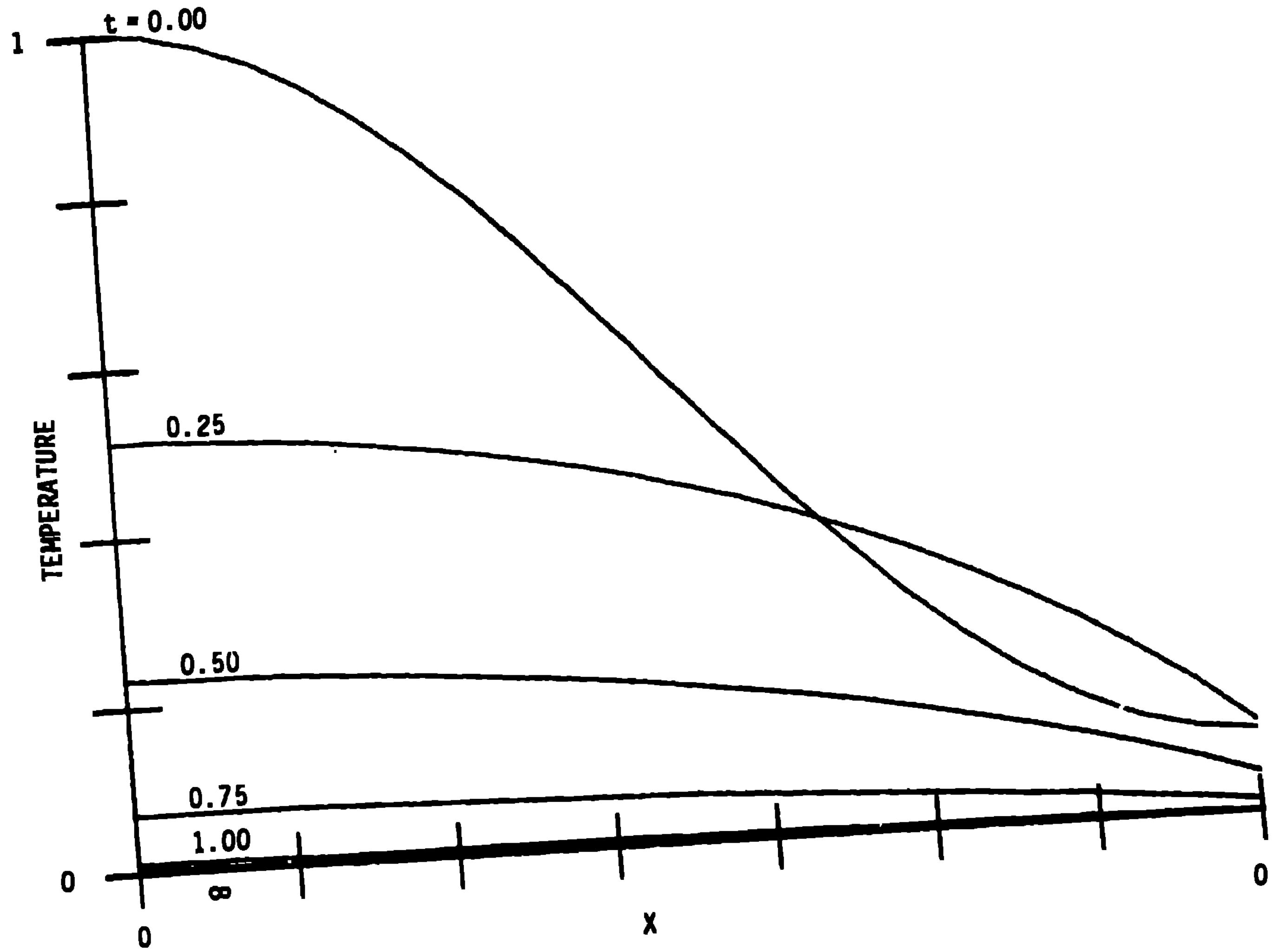


4.2a

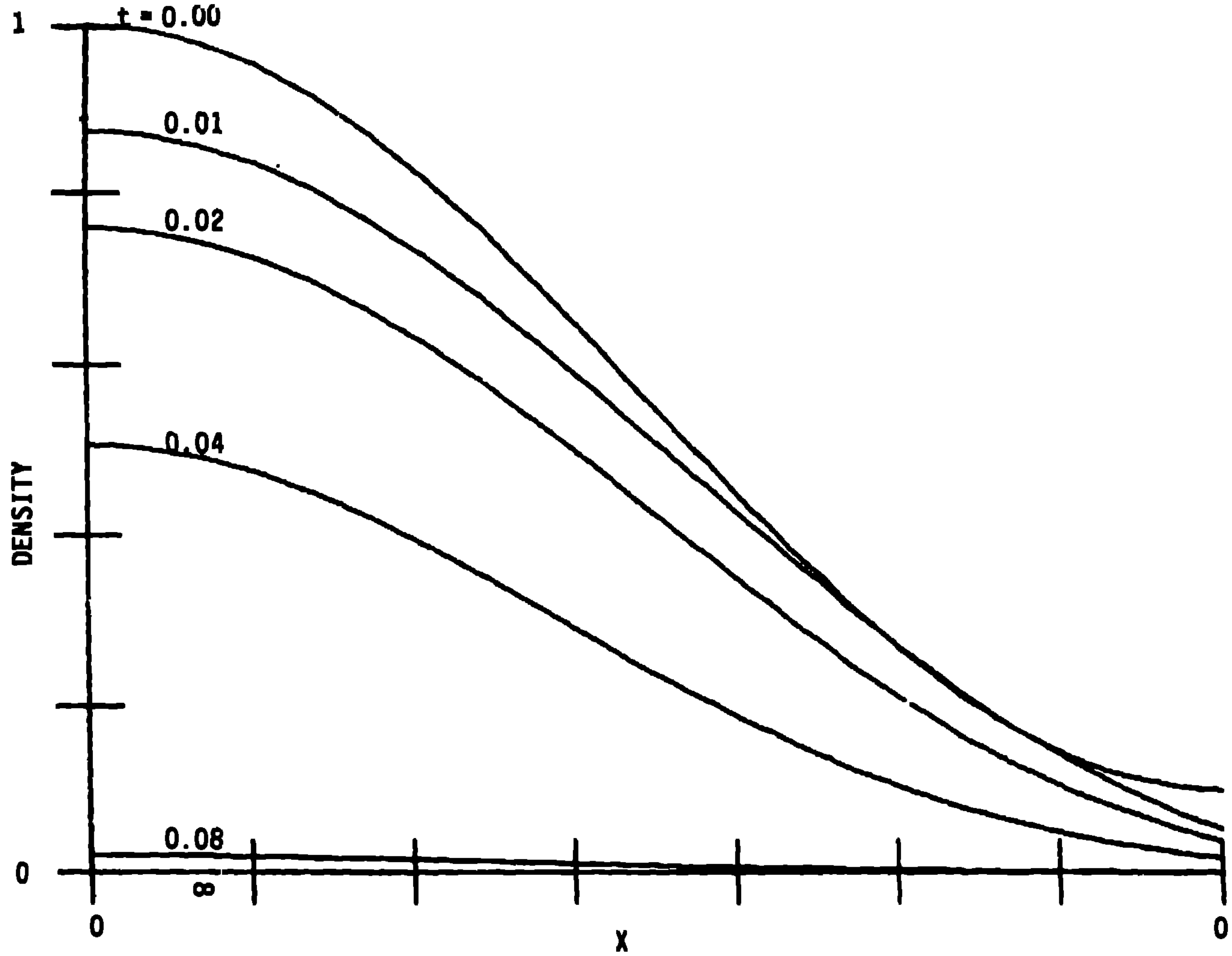

t = 0.0
0.01
0.02
0.04
0.08
∞
20
0
TEMPERATURE
0
X
1



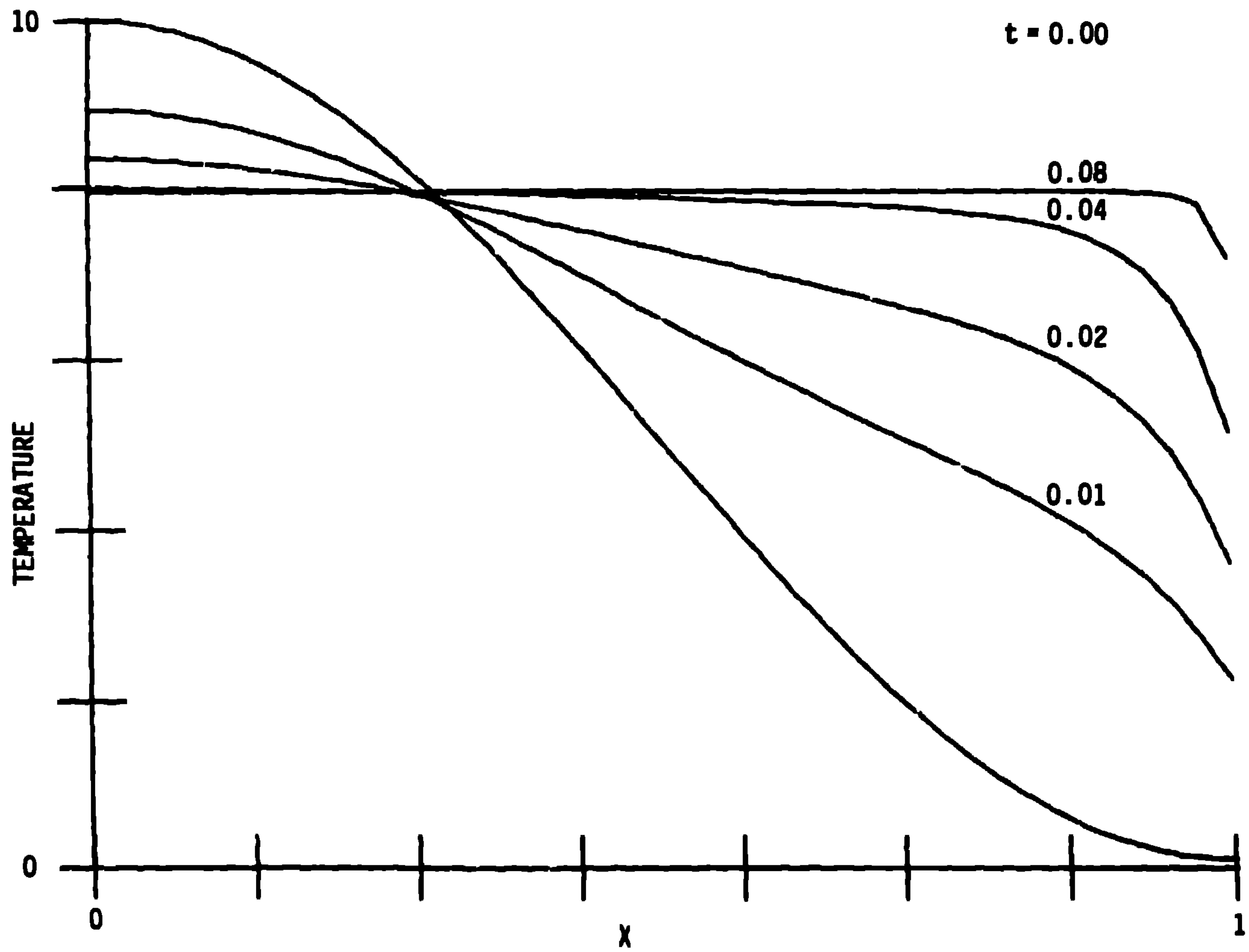
5.1a



5.16



5.2a



10
0
TEMPERATURE
0
X
1
t = 0.00
0.08
0.04
0.02
0.01