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TITLE: LOW-DIMENSIONAL BEHAVIOR OF THE PATTERN FORMATION CAHN-HILLIARD EQUATION

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LOW-DIMENSIONAL BEHAVIOR OF THE PATTERN FORMATION CAHN-HILLIARD EQUATION

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We investigate the fourth-order Cahn-Hilliard parabolic partial differential equation which describes pattern formation in phase transition. Neumann and periodic boundary conditions are considered for a domain in R^n, $1 \le n \le 3$. This equation is characterized by a negative (backward) second order diffusion and multiple steady states for the appropriate range of parameters. We establish compactness of the orbits in $H^1(\Omega)$ and convergence to some steady state. We demonstrate that the Cahn-Hilliard equation admits an intrinsic low dimensional behavior: in R^1, the number of determining modes (in a Galerkin expansion) is proportional to $L^{3/2}$; where L, the diameter of the domain, is also proportional to the number of unstable modes for the linearized equation. Similar results hold for n = 2,3.

1. INTRODUCTION

We investigate the low dimensional behavior of the Cahn-Hilliard equation with a quartic homogeneous free energy, in R^n, $1 \le n \le 3$:

$$\frac{\partial u}{\partial t} = \mathrm{div}\,[M(u)\,\nabla\,(-\Delta u + \alpha u^3 - \beta u)]$$

$$\equiv \mathrm{div}\,[M(u)\,\nabla\,J(u)] \quad \text{in } \Omega \subset R^n \;,$$

$$u(0) = u_0 \in H^2(\Omega) \;,\; \alpha > 0 \text{ and } \beta > 0 \;; \tag{1.1a}$$

the following hypotheses are made for the mobility coefficient M(u):

$$M(u) > 0 \;,\; \text{monotone non-increasing in } |u|,\; C^1$$

$$\text{and } M(u) > M(\;)\,\exp -\lambda|u| \;,\; \lambda > 0 \;; \tag{1.1b}$$

the boundary conditions on $\partial\Omega$ (boundary of the pattern cell) are either of the Neumann type or periodic (periodic cell structure):

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \;,\; \left.\frac{\partial J}{\partial \nu}\right|_{\partial\Omega} = 0 \;, \tag{1.1c}$$

or

$$u(x + Le_i t) = u(x,t) \quad 1 \le i \le n \;, \tag{1.1d}$$

L being the size of a typical pattern cell.

Eq. (1.1) is in fact a normalized form for the classical Cahn-Hilliard equation [2,5,9]:

$$\frac{\partial c}{\partial t} = \text{div}\,[M(u)\,\nabla\,(-\Delta c + b_2 c + b_3 c^2 + b_4 c^3)] \quad ,$$

$$b_2 \text{ either} > 0 \text{ or} < 0, \quad b_3 < 0,\ b_4 > 0 \quad , \tag{1.2}$$

with the same boundary conditions. As shown below (1.2) reduces to (1.1) by a simple translation $c(x,t) = u(x,t) + c^*$, c^* constant.

Eq. (1.2) is a continuum model for pattern formation resulting from phase transition. It is associated to a classical Landau-Ginzburg free energy [1]:

$$\hat{F} = \int_\Omega (\tfrac{1}{2}(\nabla\hat{c})^2 + f(\hat{c}))\,dx \quad , \quad \int_\Omega \hat{c}\,dx \equiv \int_\Omega c(x,0)\,dx = ct \quad , \tag{1.3a}$$

where the homogeneous free energy $f(c)$ is a quartic polynomial whose derivative is:

$$\frac{\partial f}{\partial c} = b_2 c + b_3 c^2 + b_4 c^3 \quad , b_3 < 0 \quad , b_4 > 0 \quad . \tag{1.3b}$$

Steady-state solutions of (1.2) are given by critical points of the <u>non-convex</u> functional F. The corresponding Euler-Lagrange equation is:

$$-\Delta\hat{c} + b_2\hat{c} + b_3\hat{c}^2 + b_4\hat{c}^3 = ct \quad , \tag{1.3c}$$

plus appropriate boundary conditions.

The influence of the homogeneous free energy function $f(c)$ appears in the sign of b_2 and the parameter B [9]:

$$B = -\frac{b_3}{(|b_2|b_4)^{\frac{1}{2}}} \quad . \tag{1.4}$$

If $b_2 \leq 0$, there is a "negative viscosity" destabilizing mechanism somewhat similar to the one observed in the Kuramoto-Sivashinsky equation for unstable flame fronts [6-8]. The <u>zero solution is unstable</u> and this regime is referred to as "unstable subspinodal." The special limit case $b_2 = 0$ is called the "spinodal regime."

If $b_2 > 0$ and $B^2 > 3$, the cubic $\frac{\partial f}{\partial c}$ defined in (1.3b) possesses two distinct extrema. If $B^2 < 3$, $b_2 > 0$, it is well known that zero is a monotonically stable attractor [5,9]. A. Novick-Cohen and L. A. Segel [9] have extensively studied the case $3 \leq B^2 < \infty$ in a one-dimensional geometry. They have specified the full set of equilibrium solutions. They have also established that for $4.5 \leq B^2 < \infty$, the basin of attraction of zero is bounded, whereas there exists at least another nontrivial equilibrium with its own basin of attraction. $B^2 = 4.5$ is the distinguished "binodal" case.

We investigate some global dynamical properties of (1.2) when $b_2 > 0$ and $B^2 > 3$, or $b_2 \leq 0$. <u>Either case reduce to the normalized equation</u> (1.1); set:

$$u(x,t) = c(x,t) - c^* \quad , \tag{1.5a}$$

where

$$c^* = -b_3/3b_4 > 0 \quad , \tag{1.5b}$$

and is such that

$$\left.\frac{\partial^3 f}{\partial c^3}\right|_{c=c^*} = 0 \quad ;$$

through the translation (1.5), the cubic $\frac{\partial f}{\partial c}$ is changed into:

$$\frac{\partial f}{\partial c} = c^* + [b_2 - \frac{1}{3}\frac{b_3^2}{b_4}]\, u + b_4\, u^3 \quad . \tag{1.6a}$$

We define

$$\alpha = b_4 > 0 \tag{1.6b}$$

$$\beta = -\, [b_2 - \frac{1}{3}\frac{b_3^2}{b_4}] \quad , \beta > 0 \quad ; \tag{1.6c}$$

indeed $B^2 > 3$, $b_2 > 0$ implies $\beta > 0$. Injecting (1.5) and (1.6) into the Cahn-Hilliard Eq. (1.2) yields the normalized form (1.1), with $M \equiv M(c^* + u)$, and $u_0 = c(x,0) - c^*$.

In Section 1, we verify boundedness of orbits in $H^1(\Omega)$ and the existence of Lyapunov functional. Although the above is implicit in the literature, <u>compactness</u> of orbits in $H^1(\Omega)$ has <u>not</u> previously been established, to our knowledge. This is done in Section 2, and enables the correct application of a classical topological dynamics theorem of Hale [4]: all orbits strongly converge in $H^1(\Omega)$ to critical points of the <u>non-convex</u> functional (1.3a).

However, the most important results are found in Section 4; we establish the intrinsically low-dimensional behavior of the Cahn-Hillard equation. Essentially, we project any orbit onto the linear manifold of the first m-eigenmodes of the biharmonic Δ^2. Suppose that the m-dimensional <u>projected</u> orbit converges to some m-dimensional fixed point; we will say that the first m-eigenmodes are <u>determining</u> if this implies convergence of the <u>infinite</u> dimensional orbit.

Following ideas developed in the Navier-Stokes context by Foias-Manley-Temam-Treve [3], we prove that for the one-dimensional Cahn-Hilliard equation:

$$m \geq ct\, L^{3/2} \quad ,$$

where L is the pattern size.

L is also proportional to the number of unstable modes of (1.1) linearized at $u = 0$; indeed the eigenvalue spectrum is:

$$\Lambda_k = \beta^2 \left(-\left(\frac{2\pi k}{\sqrt{\beta}L}\right)^4 + \left(\frac{2\pi k}{\sqrt{\beta}L}\right)^2\right) \quad , k = 0,1,2,\ldots.$$

and

$$\#\ \{\Lambda_k | \Lambda_k > 0\} = [\frac{\sqrt{\beta}}{2\pi} L] \quad ,$$

where [a] is the usual integer part of a. So for the determining modes:

$$m \geq ct\ (\#\ \text{unstable modes})^{3/2} \quad ;$$

in some heuristic sense, the impact of the nonlinearity is reflected only through the exponent ½. Similar results hold for $n = 2$ and $n = 3$, periodic boundary conditions.

To simplify the technical derivations, we restrict ourselves to $M(u) = \text{constant}$; the general case is easily disposed of, as soon as one obtains an estimate such as:

$$\overline{\lim_{t\to\infty}} \, ||u(x,t)||_{L^{\infty}(\Omega)} < K \; ;$$

then from (1.1b)

$$0 < M(0) \leq M(u) \leq M(K) \quad .$$

2. BOUNDEDNESS OF ORBITS IN $H^1(\Omega)$: THE LYAPUNOV FUNCTION

We consider the normalized problem:

$$\frac{\partial u}{\partial t} - \Delta J(u) = 0 \text{ in } \Omega \quad , \tag{2.1a}$$

$$J(u) = -\Delta u + \alpha u^3 - \beta u \quad , \alpha \text{ and } \beta > 0$$

$$u(0) = u_0 \in H^2(\Omega) \tag{2.1b}$$

with either

- periodic boundary conditions , $u(x + Le_i, t) = u(x,t)$, $1 \leq i \leq n$ (2.1c)

(L being the size of a typical pattern cell) or

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial J}{\partial \nu}\right|_{\partial\Omega} = 0 \quad . \tag{2.1d}$$

In this section, $\Omega \subset R^n$, $1 \leq n \leq 3$.

First we have the:

__Lemma 2.1__. *$\bar{u}(t) \equiv \bar{u}(0)$, where $\bar{u}(t)$ is the average $\frac{1}{|\Omega|} \int u(x,t)\ dx$ and* $|\Omega| = \text{meas}\ \Omega$.

Remark 2.2. The previous lemma implies that Poincaré-like inequalities hold, as u can be renormalized to a function of null mean value. From now on, we set

$$||u|| = (\int u^2\, dx)^{1/2} \quad ,$$

unless specified otherwise.

We now look for a Lyapunov function associated with (2.1). Multiply (4.1) by J(u) and integrate by parts over Ω. With either set of boundary conditions:

$$\int_\Omega \frac{\partial u}{\partial t} J(u)\, dx + \int_\Omega (\nabla J(u))^2\, dx = 0 \tag{2.2a}$$

and injecting the explicit form of J(u) into the first integral:

$$\frac{d}{dt}\left(\tfrac{1}{2} \int_\Omega (\nabla u)^2\, dx - \frac{\beta}{2} \int_\Omega u^2 dx + \frac{\alpha}{4} \int_\Omega u^4 dx \right) + \int (\nabla J)^2 dx = 0 \quad . \tag{2.2b}$$

Let us define V(t) as:

$$V(t) = \tfrac{1}{2} \int_\Omega (\nabla u)^2\, dx - \frac{\beta}{2} \int_\Omega u^2 dx + \frac{\alpha}{4} \int_\Omega u^4 dx \quad . \tag{2.3}$$

Then (2.2b) implies:

$$\frac{d}{dt} V(t) \leq 0 \quad . \tag{2.4}$$

To establish that V(t) is a Lyapunov function, we must show the boundedness of orbits in $H^1(\Omega)$ and that V(t) is <u>bounded</u> <u>from</u> <u>below</u> in $H^1(\Omega)$. Remark that:

$$V(t) = \tfrac{1}{2} \int_\Omega (\nabla u)^2 dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx - \frac{\beta^2}{4\alpha} |\Omega| \quad ; \tag{2.5}$$

now

$$V(t) \leq V(0) \quad , \tag{2.6}$$

so

$$\tfrac{1}{2} \int_\Omega (\nabla u)^2\, dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx \leq \tfrac{1}{2} \int_\Omega (\nabla u_0)^2\, dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u_0^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx \quad . \tag{2.7}$$

This proves the

Theorem 2.3. $\overline{\lim_{t\to\infty}}\, ||\nabla u(t)|| \leq F(u_0)$, *where*

$$F(u_0) = (||\nabla u_0^2|| + 2 \int_\Omega (\frac{\sqrt{\alpha}}{2} u_0^2 - \frac{\beta}{2\sqrt{\alpha}})^2 dx)^{\frac{1}{2}} \quad . \tag{2.8}$$

<u>*Corollary 2.4*</u>. $\overline{\lim_{t\to\infty}} ||u||_{L^4}$ *is bounded.*

<u>Proof</u>. Use the continuous imbedding

$$H^1(\Omega) \hookrightarrow L^4(\Omega) \quad , \; n \le 4$$

or specifically Eq. (2.7), together with Poincaré's inequality.

<u>*Corollary 2.5*</u>. *V(t) is a continuous, bounded from below, Lyapunov functional on* $H^1(\Omega)$.

<u>Remark 2.6</u>. All of the above results are valid if we consider the more general equation (1.1) with the coefficient of diffusion $M(u)$ given as in (1.1b). Indeed:

$$\frac{\partial u}{\partial t} - \text{div } M(u) \nabla J(u) = 0 \quad ;$$

multiplying by $J(u)$ and integrating over Ω:

$$\int_\Omega \frac{\partial u}{\partial t} J(u) \, dx + \int_\Omega M(u) \, (\nabla J)^2 \, dx = 0 \quad ,$$

and we still have

$$\frac{d}{dt} V(t) \le 0 \quad ,$$

with $V(t)$ same as in (2.3).

3. ASYMPTOTIC BEHAVIOR OF ORBITS.

We wish to establish some kind of convergence of the orbits $u(x,t)$ to the critical manifold M of fixed points $\hat{u}(x)$ of:

$$-\Delta\hat{u} + \alpha \, \hat{u}^3 - \beta \, \hat{u} = \gamma \tag{3.1a}$$

$$\int_\Omega \hat{u} \, dx = |\Omega| \bar{u}(0) \tag{3.1b}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \text{ or periodic boundary conditions} \quad . \tag{3.1c}$$

To apply classical topological dynamics results of Hale [4], we first need the relative compactness of orbits $u(t)$ in $H^1(\Omega)$:

Theorem 3.1. $\overline{\lim_{t\to\infty}} ||D^2u||$ *is bounded*[1]*, for either periodic boundary conditions (2.1c) or Neumann conditions (2.1d) if* $\Omega \subset R^1$*; and for periodic boundary conditions if* $\Omega \subset R^2$ *or* R^3.

The proof is technical and will be outlined below. Theorem 3.1 ensures the relative compactness of the orbit $u(t)$ in $H^1(\Omega)$; hence, the ω-limit set associated to u_0 is nonempty, compact, invariant and connected. Using a classical theorem for such flows with Lyapunov functions [4], namely that $V(t)$ is constant on $\omega(u_0)$, we deduce:

Corollary 3.2. As $t \to \infty$, $\lim \text{dist } |u(x,t) - M| = 0$ in $H^1(\Omega)$, *for either boundary conditions if* $\Omega \subset R^1$*, and for periodic boundary conditions if* $\Omega \subset R^2$ *or* R^3.

Remark 3.3. Problem (3.1) usually admits multiple solutions, whether one considers β or $L = \text{diam } \Omega$ as a bifurcation parameter [9].

Proof of Theorem 3.1. Multiply (2.1) by $\dfrac{\partial^4}{\partial x_1^{2\delta_1} \cdots \partial x_n^{2\delta_n}} u$, integrate by parts and take the sumation over all $\delta = (\delta_1, \ldots, \delta_n)$ such that $|\delta| = 2$; we get:

$$\frac{1}{2} \frac{d}{dt} ||D^2u||^2 + ||D^4u||^2 - \beta||D^3u||^2 = \sum_{|\delta|=2} \alpha \int \Delta u^3 \, D^{2\delta}u \, dx$$

$$= \sum_{|\delta|=2} \left(6\alpha \int u|\nabla u|^2 D^{2\delta}u \, dx + 3\alpha \int u^2 \Delta u \, D^{2\delta}u \, dx\right) \quad . \tag{3.2}$$

Apply Cauchy-Schwartz and Cauchy-Young's inequalities to the R.H.S. of (3.2):

$$\frac{1}{2} \frac{d}{dt} ||D^2u||^2 + (1-\varepsilon) ||D^4u||^2 \le \beta ||D^3u||^2 + C(\varepsilon) \int u^2(\nabla u)^4 \, dx$$

$$+ C(\varepsilon) \int u^4 (\Delta u)^2 \, dx \quad ; \tag{3.3}$$

<u>from now on</u> $C(\varepsilon)$ <u>will be a generic symbol for any constant depending upon</u> ε.

We will estimate:

$$J_1 = \int u^2 (\nabla u)^4 \, dx \quad , \tag{3.4}$$

$$J_2 = \int u^4 (\Delta u)^2 \, dx \quad . \tag{3.5}$$

(1) For brevity, we set $||D^ku||^2 = \sum_{|\alpha|=k} ||D^\alpha u||^2$.

We will need the Agmon inequalities (for functions periodic and/or with zero mean value):

$$\|u(t)\|_{L^\infty} \le \begin{cases} \gamma_1 \|u(t)\|^{\frac{1}{2}} \, \|\nabla u(t)\|^{\frac{1}{2}} , & \text{if } n = 1 , \\ \gamma_2 \|u(t)\|^{\frac{1}{2}} \|\Delta u(t)\|^{\frac{1}{2}} , & \text{if } n = 2 \\ \gamma_3 \|u(t)\|^{\frac{1}{4}} \, \|\Delta u(t)\|^{\frac{3}{4}} , & \text{if } n = 3 . \end{cases} \tag{3.6}$$

We also need the following general interpolation inequalities:

$$\|D^{k+1}u\| \le \|D^{k-1}u\|^{1/3} \, |D^{k+2}u\|^{2/3} \tag{3.7}$$

$$\|D^k u\| \le \|D^{k-1}u\|^{\frac{1}{2}} \|D^{k+1}u\|^{\frac{1}{2}} \tag{3.8}$$

Also, as $H^{\frac{1}{2}} \hookrightarrow L^4$ $(n = 2)$ or $H^{\frac{3}{4}} \hookrightarrow L^4$ $(n = 3)$, we will need:

$$\|Du\|^4_{L^4} \le \|Du\|^3 \, \|D^3u\| \quad , \; n = 2 \;\; ; \tag{3.9a}$$

$$\|Du\|^4_{L^4} \le \|Du\|^{5/2} \, \|D^3u\|^{3/2} \;, \; n = 3 \;\; ; \tag{3.9b}$$

which are obtained by interpolation of $H^{\frac{1}{2}}$ (resp. $H^{\frac{3}{4}}$) between L^2 and H^2. We will give explicit technical details only for $n = 2$. The case $n = 1$ and $n = 3$ are similar.

In (3.3), we first consider the term $\beta\|D^3u\|^2$; from (3.7) and using Cauchy-Young's inequality with $p = 3/2$, $q = 3$:

$$\begin{aligned} \|D^3u\|^2 &\le \|D^4u\|^{4/3} \, \|Du\|^{2/3} \, \| \le \varepsilon \, \|D^4u\|^2 + C(\varepsilon) \, \|Du\|^2 \\ &\le \varepsilon \, \|D^4u\|^2 + C(\varepsilon) \quad , \end{aligned} \tag{3.10}$$

since $\overline{\lim_{t\to\infty}} \|\nabla u\| \le F(u_0)$ (Theorem 2.3).

Now estimate J_1 in (3.4):

$$\int u^2(\nabla u)^4 dx < \|u\|^2_{L^\infty} \, \|\nabla u\|^4_{L^4} \quad ;$$

using Agmon's inequalities (3.6) and the interpolation inequality (3.9a):

$$J_1 < Ct \, \|u\| \, \|D^2u\| \, \|Du\|^3 \, \|D^3u\| \quad ,$$

and from Theorem 2.3:

$$J_1 < Ct \, \|D^2u\| \, \|D^3u\| < Ct \, \|D^3u\|^2$$

(using Poincaré's inequality) and

$$J_1 < \varepsilon \, \|D^4u\|^2 + C(\varepsilon) \quad , \tag{3.11}$$

following (3.10).

Now estimate J_2 in (3.5):

$$\int u^4 (\Delta u)^2\, dx \leq ||\Delta u||^2_{L^\infty}\, ||u^4||^4_{L^4} \quad ;$$

using Agmon's inequalities (3.6):

$$J_2 \leq Ct\ ||\Delta u||\ ||D^4 u||\ ||u^4||^4 \leq Ct\ ||D^2 u||\ ||D^4 u|| \quad ,$$

(using Corollary 2.4); now using the interpolation inequality (3.8):

$$J_2 \leq Ct\ ||Du||^{\frac{1}{2}}\ ||D^3 u||^{\frac{1}{2}}\ ||D^4 u|| \leq Ct\ ||D^3 u||^{\frac{1}{2}}\ ||D^4 u|| \quad ;$$

but from the interpolation inequality (3.7):

$$||D^3 u|| \leq ||Du||^{1/3}\ ||D^4 u||^{2/3} \quad ;$$

so:

$$J_2 \leq Ct\ ||Du||^{1/6}\ ||D^4 u||^{4/3} \quad ,$$

and using Cauchy-Young's inequality with $p = 3/2$, $q = 3$:

$$J_2 \leq \varepsilon\ ||D^4 u||^2 + C(\varepsilon)\ ||Du||^{\frac{1}{2}} \quad ,$$

$$J_2 \leq \varepsilon\ ||D^4 u||^2 + C(\varepsilon) \quad . \tag{3.12}$$

We now collect all terms in Eq. (3.3), applying (3.10, 3.11, 3.12):

$$\frac{1}{2} \frac{d}{dt} ||D^2 u||^2 + (1 - 3\varepsilon - \beta\varepsilon)\ ||D^4 u||^2 < C(\varepsilon) \quad . \tag{3.13}$$

We conclude with the help of Poincaré's inequality and Gronwall's Lemma, that:

$$\overline{\lim_{t\to\infty}}\ ||D^2 u|| < \infty \quad . \qquad \square$$

4. NUMBER OF DETERMINING MODES

This section gives our main result, namely an upper bound of the number of determining modes for any solution of the Cahn-Hilliard equation (2.1) with periodic boundary conditions. This bound is formulated in terms of L. Although we give the detailed derivation for space dimension $n = 1$, analogue results can easily be derived for $n = 2$ and $n = 3$.

Consider u,v two solutions of (2.1), corresponding to two initial data (in $H^2(\Omega)$); set $w = u-v$. Due to the periodicity of u,v, we can use a Fourier mode decomposition of w and set:

$$P_m w(x,t) = \sum_{|k| \leq m} w_k(t) \exp\frac{2i\pi}{L} k.x \tag{4.1}$$

where $k \in Z^n$, and $w_k(t)$ is the k^{th} Fourier coefficient of $w(x,t)$. We will also use:

$$Q_m\, w(x,t) = (I - P_m) w(x,t) \quad . \tag{4.2}$$

Definition 4.1. We say that the first m Fourier modes of $w = u-v$ are determining if:

$$\lim_{t\to\infty} ||P_m (u(t) - v(t))|| = 0 \rightarrow \lim_{t\to\infty} ||u(t) - v(t)|| = 0 \quad . \qquad (4.3a)$$

Remark 4.2. For Neumann boundary conditions (2.1d), we use the appropriate eigenfunctions of (Δ^2) as a Galerkin basis in (4.1 - 4.2).

Remark 4.3. If Ξ is a compact positive invariant set under the semi-flow defined in Section 3, then from (4.3) we deduce:

$$\lim_{t\to\infty} \text{dist} =(P_m u(t), P_m \Xi) = 0 \rightarrow \lim_{t\to\infty} \text{dist}(u(t), \Xi) = 0 \quad ,$$

since $v(t) \in \Xi$ for all times if $v(0) \in \Xi$.

In particular, if $u \equiv u^*$, where u^* is some equilibrium solution belonging to the set of M of fixed points (cf. Eq. (3.1), then:

$$\lim_{t\to\infty} ||P_m u(t) - P_m u^*|| = 0 \rightarrow \lim_{t\to\infty} ||u(t) - u^*|| = 0 \quad ; \qquad (4.3b)$$

if the projection of the orbit converges to some (projected) fixed point, the same is true of the infinite-dimensional orbit.

The main result of this section is stated for space dimension $n = 1$; with $\Omega = [0,L]$ and periodic boundary conditions:

Theorem 4.4. The first m Fourier modes are determining if

$$m + 1 \geq K L^{3/2} \quad , \qquad (4.4)$$

where K is some constant depending on α,β and ζ_0, with initial values $||\nabla u(0)|| \leq \zeta_0$.

Proof of Theorem 4.4. For sake of brevity, in the sequel, we will denote $q_m \equiv Q_m w$, $p_m \equiv P_m w$. Now, if u,v are two solutions, w satisfy the following equation:

$$\frac{\partial w}{\partial t} + \Delta(\Delta w + \beta w - \alpha[u^2 + uv + v^2]w) = 0 \quad . \qquad (4.5a)$$

Multiplying by q_m and integrating:

$$\frac{1}{2} \frac{d}{dt} ||q_m||^2 + ||\Delta q_m||^2 - \beta ||\nabla q_m||^2 - \alpha \int [u^2 + uv + v^2] w \Delta q_m dx = 0 \qquad (4.5b)$$

But $w = q_m + p_m$, and so by Hölder's inequality:

$$\int (u^2 + uv + v^2) w \Delta q_m dx$$

$$\leq ||u^2 + uv + v^2||_{L^\infty} (||p_m|| + ||q_m||) ||\Delta q_m|| \qquad (4.6)$$

and

$$\frac{1}{2}\frac{d}{dt}\,||q_m||^2 + \frac{1}{||q_m||^2}\{||\Delta q_m||^2 - \beta\,||\nabla q_m||^2$$

$$- \alpha||u^2 + uv + u^2||_{L^\infty}\,||\Delta q_m||\,\,||q_m||\}\,\,||q_m||^2$$

$$\leq \alpha\,||u^2 + uv + v^2||_{L^\infty}\,||\Delta q_m||\,\,||p_m|| \quad . \tag{4.7}$$

<u>We must prove that</u> $||p_m|| \to 0$ <u>implies</u> $||q_m|| \to 0$. This will be completed by verifying the three assumptions of the generalized Gronwall's Lemma 4.1 of [3]. We recall this Lemma:

Let $\xi(t)$ be an absolutely continuous nonnegative function on $(0,\infty)$ such that

$$\frac{d\xi}{dt} + A(t)\xi \leq B(t) \quad \text{a.e. on } (0,\infty) \quad ,$$

where $A(t)$ is a locally integrable function on $(0,\infty)$ satisfying for some T, $0 < T < \infty$:

$$\liminf_{t\to\infty} \int_t^{t+T} A\,ds = \gamma > 0 \tag{H1}$$

$$\limsup_{t\to\infty} \int_t^{t+T} A^-\,ds = \Gamma < \infty \quad , \tag{H2}$$

where $A^- = \max\,(-A, 0)$ and $B(t)$ is a measurable function on $(0,\infty)$ such that

$$B(t) \to 0 \quad , \; t \to \infty \quad , \tag{H3}$$

then

$$\xi(t) \to 0 \text{ as } t \to \infty \quad .$$

(Here, we set $\xi(t) \equiv ||q_m(t)||^2$.) We define:

$$A_m(t) = 2\,\frac{||\Delta q_m|| - \beta\,||\nabla q_m||^2}{||q_m||^2} - 2\alpha\,\frac{||u^2 + uv + v^2||_{L^\infty}}{||q_m||}\,||\Delta q_m|| \tag{4.8}$$

$$B_m(t) = 2\alpha\,||u^2 + uv + v^2||_{L^\infty}\,||\Delta q_m||\,\,||p_m|| \quad , \tag{4.9}$$

$$\rho_m(t) = \frac{||\Delta q_m||^2}{||q_m||^2} \quad , \; \tilde{\rho}_m(t) = \frac{1}{T}\int_t^{t+T} \rho_m(s)\,ds \quad , \tag{4.10}$$

$$R(u,v) = \alpha||u^2 + uv + v^2||_{L^\infty} \quad . \tag{4.11}$$

Inequality (4.7) now can be rewritten in a more compact way:

$$\frac{d}{dt} ||q_m||^2 + A_m(t) \; ||q_m||^2 \leq B_m(t) \quad . \tag{4.12}$$

We first verify Hypothesis (H1) from the generalized Gronwall's Lemma:

$$A_m(t) \geq \frac{2||\Delta q_m||^2}{||q_m||^2} - \frac{2\beta||\Delta q_m||}{||q_m||} - 2\; R(u,v)\; \frac{||\Delta q_m||}{||q_m||}$$

$$= 2\;\rho_m(t) - 2\;\beta\;\rho_m(t)^{\frac{1}{2}} - 2\;R(u,v)\;\rho_m(t)^{\frac{1}{2}} \quad . \tag{4.13}$$

From (4.13):

$$\frac{1}{T}\int_t^{t+T} A_m(s)\;ds \geq 2\;\tilde{\rho}_m(t) - 2\;\beta\;\tilde{\rho}_m(t)^{\frac{1}{2}} - \frac{2}{T}\int_t^{t+T} R(u,v)\;\rho_m(s)^{\frac{1}{2}}\;ds$$

$$\geq 2\;\tilde{\rho}_m(t) - 2\;\beta\;\tilde{\rho}_m(t)^{\frac{1}{2}} - 2\left(\frac{1}{T}\int_t^{t+T} R(u,v)^2\;ds\right)^{\frac{1}{2}}\tilde{\rho}_m(t)^{\frac{1}{2}}$$

$$= 2\;\tilde{\rho}_m(t)^{\frac{1}{2}}\,[\tilde{\rho}_m(t)^{\frac{1}{2}} - \beta - (\frac{1}{T}\int_t^{t+T} R(u,v)^2\;ds)^{\frac{1}{2}}] \quad , \tag{4.14}$$

where we use a classical interpolation inequality for $||\nabla q_m||^2$ and Jenssen's inequality. From (4.14), a sufficient condition for (H1) is:

$$\tilde{\rho}_m(t)^{\frac{1}{2}} \geq \beta + (\frac{1}{T}\int_t^{t+T} R(u,v)^2\;ds)^{\frac{1}{2}} \quad ; \tag{4.15}$$

but

$$\tilde{\rho}_m(t) \geq E_{m+1} \quad , \tag{4.16}$$

where E_{m+1} is the $(m+1)^{th}$ eigenvalue of the biharmonic; $E_{m+1} = (\frac{2\pi(m+1)}{L})^4$. Then a sufficient condition for hypothesis (H1) is:

$$\frac{4\pi^2(m+1)^2}{L^2} > \beta + 4\alpha\;[\frac{1}{T}\int_t^{t+T} \max\;(||u^2||^2_{L^\infty}\;,\;||v^2||^2_{L^\infty}\;ds]^{\frac{1}{2}} \quad . \tag{4.17}$$

We will further elaborate on (4.17). But we first verify Hypothesis (H2) and (H3) from the generalized Gronwall's Lemma. To verify (H2), notice that (4.14) implies by the Cauchy-Young inequality:

$$\frac{1}{T}\int_t^{t+T} A_m(s)\;ds \geq 2\;\tilde{\rho}_m(t) - 2\;\beta\;\tilde{\rho}_m(t)^{\frac{1}{2}} - \tilde{\rho}_m(t) - \lim_{t\to\infty} R(u,v)^2 \quad ; \tag{4.18}$$

(H2) is satisfied as soon as

$$\tilde{\rho}_m(t) \geq 4\;\beta^2 \tag{4.19}$$

which is implied by (4.16) and (4.17). To verify (H3), remember that $R(u,v)$ and $||\Delta q_m||$ are uniformly bounded in time (cf., Section 3); moreover, $||p_m(t)|| \to 0$ from the very hypothesis of theorem 4.4.

We now further explicit the remaining sufficient condition (4.17). Using (Lemma 2.1), namely that

$$\bar{u}(t) \equiv \bar{u}(0) \quad ,$$

the continuous injection of $H^1(\Omega)$ into $L^\infty(\Omega)$ can be sharpened as:

$$||u||_{L^\infty} \le \sqrt{L}\ ||\nabla u||_{L^2} + \bar{u}(0) \quad . \tag{4.20}$$

Then:

$$\left(\frac{1}{T}\int_t^{t+T} \max\ (||u^2||^2_{L^\infty}\ ,\ ||v^2||^2_{L^\infty}\ ds\right)^{\frac{1}{2}}$$

$$\le \max\ (\overline{\lim_{t\to\infty}}\ ||u||^2_{L^\infty}\ ,\ \overline{\lim_{t\to\infty}}\ ||v||^2_{L^\infty})$$

$$\le \max\ ((\sqrt{L}\ F(u_0) + \bar{u}(0))^2\ ,\ (\sqrt{L}\ F(v_0) + \bar{v}(0))^2) \quad , \tag{4.21}$$

where we have used Theorem 2.3, i.e., $\overline{\lim_{t\to\infty}}\ ||\nabla u(t)|| \le F(u_0)$. Then for m and L large enough, (4.17) is equivalent to:

$$\frac{4\pi^2(m+1)^2}{L^2} \sim Ct(\alpha,\beta,u_0,v_0)\ L \quad , \tag{4.22a}$$

$$m + 1 \sim Ct(\alpha,\beta,\zeta_0)\ L^{3/2} \quad , \tag{4.22b}$$

where we have taken both $||\nabla u(0)||$ and $||\nabla v(0)|| < \zeta_0$. □

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References

1. K. Binder, Z. Physik 267 (1974), 213.
2. J. W. Cahn and J. E. Hilliard, J. Chem. Phys. 28 (1958), 258.
3. C. Foias, O. P. Manley, R. Temam and Y. M. Treve, "Asymptotic analysis of the Navier-Stokes equations," Physica 9D (1983), 157-188.
4. J. K. Hale, "Dynamical Systems and Stability," J. Math. Anal. Appl. 26 (1969), 39-59.
5. J. S. Langer, Annals of Physics (N.Y.) 65 (1971), 53.

6. B. Nicolaenko and B. Scheurer, "Remarks on the Kuramoto-Sivashinsky Equation," Physica 12D (1984), 391-395.
7. B. Nicolaenko, B. Scheurer and R. Temam, "Some Global Dynamical Properties of the Kuramoto-Sivashinsky Equations: Nonlinear Stability and Attractors," Los Alamos National Laboratory report LA-UR-84-2326 (1984).
8. B. Nicolaenko, B. Scheurer and R. Temam, "Quelques proprietes des attracteurs pom l'équation de Kuramoto-Sivashinsky," C. R. Acad. Sc. Paris 298 (1984), 23-25.
9. A. Novick-Cohen and L. A. Segel, "Nonlinear Aspects of the Cahn-Hilliard Equation," Physics D (1984), to appear.