

NUMERICAL TREATMENT OF LINEARIZED EQUATIONS DESCRIBING
INHOMOGENEOUS COLLISIONLESS PLASMAS

H. Ralph Lewis

Los Alamos Scientific Laboratory, P. O. Box 1663

Los Alamos, New Mexico 87545, U. S. A.

I. INTRODUCTION

There is considerable current interest in the initial-value problem for the linearized equations which describe small departures from equilibrium of a fully ionized plasma in which one or more of the particle species can be treated as collisionless. Present-day research in controlled thermonuclear fusion requires information about the stability of such systems and the effects of phase mixing in them; this is also true in other fields, such as space physics, in which the physics of collisionless plasmas plays a role. During any specified period of time, the collisionless description for a particular particle species in an experimental plasma applies if the temperature of the species is sufficiently high. The linearized equations for spatially inhomogeneous plasmas in which there is a collisionless species are difficult to solve, even computationally, because all three velocity components and at least one spatial coordinate must be considered as independent variables in the analysis. This means that the equations are a system of coupled integrodifferential equations in which there are at least five independent variables--time, three velocity components, and at least one spatial coordinate. Recently, progress has been made in the formulation of the problem in terms of a dispersion matrix, and applications of the formulation to interesting equilibria with one nonignorable coordinate have been made. When there is one nonignorable coordinate in the equilibrium, only that spatial coordinate appears as an independent variable in the system of integrodifferential equations. The formulation is in terms of a description of the three-dimensional equilibrium motion of particles which is obtained by using an equivalent one-dimensional potential. Integrals with respect to time arise which extend over the times appropriate for the equivalent one-dimensional problem; for orbits which are trapped in the equivalent one-dimensional potential, the integrals extend over the bounce periods in the one-dimensional potential and not over the infinite time history of the equilibrium three-dimensional orbits. This approach has been described in the context of a general discussion of the initial-value problem for linearized equations which describe plasma systems in which there is a collisionless species.¹ Applications of the general formalism have been made to the stability of a plasma column within the

framework of the Vlasov-fluid model² and to the stability of large-amplitude Bernstein-Greene-Kruskal equilibria.³ The basic approach has also been used independently in the context of the Vlasov-fluid model to study the stability of a rotating theta pinch,⁴ and to investigate the effects of resonant particles on kinetic stabilization in screw pinches.⁵ Additional work is in progress.

In Section II, the basic linearized equations are presented in a general context, specialization for one nonignorable coordinate is indicated, and a formulation for numerical work is introduced. Numerical aspects of the problem are discussed in Section III, including choice of matrix representation and methods of solution. Some concluding remarks are given in Section IV.

II. GENERAL THEORETICAL FRAMEWORK

We consider a plasma which consists of one or more collisionless particle species which are governed by a linearized Boltzmann equation for each collisionless species s ,

$$\left(\frac{\partial}{\partial t} + iL_s\right)f_s^{(1)} = U_s\phi^{(1)}; \quad (1)$$

and we assume that the plasma can be described by these equations and a set of field equations of the form

$$K\phi^{(1)} = \sum_s \int d^3v J_s(\underline{r}, \underline{v}) f_s^{(1)}(\underline{r}, \underline{v}, t). \quad (2)$$

The quantities L_s , U_s , and K are linear operators, and $f_s^{(1)}$ is the perturbation of a single-particle distribution function f_s about an equilibrium distribution function $f_s^{(0)}$:

$$f_s(\underline{r}, \underline{v}, t) = f_s^{(0)}(\underline{r}, \underline{v}) + f_s^{(1)}(\underline{r}, \underline{v}, t). \quad (3)$$

The quantity $\phi^{(1)}$ is the perturbation of an array ϕ of potential functions about an equilibrium array $\phi^{(0)}$:

$$\phi(\underline{r}, t) = \phi^{(0)}(\underline{r}) + \phi^{(1)}(\underline{r}, t) . \quad (4)$$

The symbols \underline{r} and \underline{v} denote position and velocity vectors.

A simple example of equations of this form is the set of linearized equations for a one-dimensional electron gas in a background of immobile ions of number density $n_0(x)$:

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{e}{m} \frac{d\phi^{(0)}}{dx} \frac{\partial}{\partial v} \right) f^{(1)} = - \frac{e}{m} \frac{\partial f^{(0)}}{\partial v} \frac{\partial \phi^{(1)}}{\partial x} , \quad (5)$$

$$\frac{\partial^2 \phi^{(1)}}{\partial x^2} = 4\pi e \int dv f^{(1)}(x, v, t) . \quad (6)$$

The symbols x and v denote the one-dimensional position and velocity variables. The quantity ϕ represents a single potential function, the scalar potential for the electric field; $\phi^{(0)}$ is a function of x only which is related to $f^{(0)}$ by

$$\frac{\partial^2 \phi^{(0)}}{\partial x^2} = 4\pi e \left[\int dv f^{(0)}(x, v) - n_0(x) \right] . \quad (7)$$

The electron mass is m , the electron charge is $-e$, and the ion charge is e .

An example of equations of the form of (1)-(2) which are useful for describing a plasma in a magnetic field is given by the Vlasov fluid model.⁵ This model is a low-frequency model for an ion-electron plasma in which the ions are treated as collisionless and the electrons are treated as a massless, pressureless fluid. The linearized equations are

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}_{\underline{r}} + \frac{e}{M} \left(\underline{E}^{(0)} + \frac{1}{c} \underline{v} \times \underline{B}^{(0)} \right) \cdot \underline{\nabla}_{\underline{v}} \right] f^{(1)} = - e \frac{df^{(0)}(\underline{r})}{d\underline{r}} \cdot \underline{v} \cdot \underline{E}^{(1)} , \quad (8)$$

$$\frac{1}{4\pi} [(\nabla \times \underline{B}^{(0)}) \times \underline{B}^{(1)} + (\nabla \times \underline{B}^{(1)}) \times \underline{B}^{(0)}] - en^{(0)} \underline{E}^{(1)} \\ = e \int d^3v [\underline{E}^{(0)} + \frac{1}{c} \nabla \times \underline{B}^{(0)}] f^{(1)}. \quad (9)$$

Here, e and M are the ion charge and mass, respectively; c is the speed of light; \underline{E} and \underline{B} are the electric and magnetic fields, respectively; f is the ion single-particle distribution function; the equilibrium distribution function is assumed to be a function of the energy ϵ only; and ϵ and n_0 are defined by

$$\epsilon = \frac{1}{2} Mv^2 + e\phi^{(0)}(\underline{r}), \quad (10)$$

$$n^{(0)}(\underline{r}) = \int d^3v f^{(0)}(\epsilon). \quad (11)$$

The array of perturbation potentials in (8)-(9) is the set of components of a displacement vector $\underline{\xi}(\underline{r}, t)$ from which $\underline{E}^{(1)}$ and $\underline{B}^{(1)}$ are derived:

$$\underline{E}^{(1)} = -\frac{1}{c} \frac{\partial \underline{\xi}}{\partial t} \times \underline{B}^{(0)} - \nabla(\underline{E}^{(0)} \cdot \underline{\xi}), \quad (12)$$

$$\underline{B}^{(1)} = \nabla \times (\underline{\xi} \times \underline{B}^{(0)}), \quad (13)$$

where

$$\underline{\xi} \cdot \underline{B}^{(0)} = 0. \quad (14)$$

For some examples of equations (1)-(2), it can be useful to replace the perturbation distribution function $f^{(1)}$ by an auxiliary function g_s which is a linear functional of the perturbation potentials:

$$g_s = f_s^{(1)} - P_s \hat{f}^{(1)}, \quad (15)$$

where P_g is a linear operator. The equations for the auxiliary functions and the perturbation potentials are of the same form as (1)-(2):

$$\left(\frac{\partial}{\partial t} + iL_g\right)g_a = W_g \phi^{(1)}, \quad (16)$$

$$\Lambda \phi^{(1)} = \int_g d^3v J_g(\underline{r}, \underline{v}) g_a(\underline{r}, \underline{v}, t), \quad (17)$$

where

$$\Lambda = K - \int_g d^3v J_g(\underline{r}, \underline{v}) P_g. \quad (18)$$

The utility of such a transformation of dependent variables is that the field operator Λ which appears in (17) can be chosen arbitrarily while preserving the form of the equations. In particular, the operator on the left hand side of (16) is unchanged by this transformation. The introduction of auxiliary functions in this way can be advantageous in numerical approximation schemes, a point to which we return later. Of course, the original equations, (1)-(2), are an example of (16)-(17). The original field operator, K , involves time differentiation for some physical systems of interest; an example is the case of the Vlasov-fluid model as indicated in (9). For these systems it is possible to introduce auxiliary functions such that Λ does not involve t or the operator $\partial/\partial t$. Henceforth, we consider (16)-(17) as the basic equations and assume that Λ does not involve t or $\partial/\partial t$.

It is convenient to consider the solution of (16)-(17) for the evolution of g_a and the perturbation potentials in terms of Laplace transforms. We denote the Laplace transform of a function $h(t)$ by $\hat{h}(\omega)$, where

$$h(t) = \frac{1}{2\pi} \int_C d\omega e^{-i\omega t} \hat{h}(\omega), \quad \hat{h}(\omega) = \int_0^\infty dt e^{i\omega t} h(t), \quad (19)$$

and C is a suitable Bromwich contour. For the physical applications which we envision, the linear operator W_g may involve the time differentiation operator $\partial/\partial t$, but time does not occur in W_g in any other way. Therefore, it is appropriate to define $\hat{W}_g(\omega)$ as the result of substituting $-i\omega$ for $\partial/\partial t$ in W_g . The solution of (16)-(17) for the Laplace transforms $\hat{g}_a(\omega)$ and $\hat{\phi}^{(1)}(\omega)$ is

$$\hat{\phi}^{(1)}(\omega) = -iD^{-1}(\omega) \int_{\mathcal{G}} d^3v J_{\mathcal{G}} [L_{\mathcal{G}} - \omega]^{-1} [\phi_{\mathcal{G}}(\omega) + g_{\mathcal{G}}(0)] , \quad (20)$$

$$\begin{aligned} \hat{g}_{\mathcal{G}}(\omega) = & - [L_{\mathcal{G}} - \omega]^{-1} \hat{W}_{\mathcal{G}}(\omega) D^{-1}(\omega) \int_{\mathcal{G}'} d^3v J_{\mathcal{G}'} [L_{\mathcal{G}'} - \omega]^{-1} [\phi_{\mathcal{G}'}(\omega) + g_{\mathcal{G}'}(0)] \\ & - i [L_{\mathcal{G}} - \omega]^{-1} [\phi_{\mathcal{G}}(\omega) + g_{\mathcal{G}}(0)] , \end{aligned} \quad (21)$$

where

$$D(\omega) = \Lambda + i \int_{\mathcal{G}} d^3v J_{\mathcal{G}} [L_{\mathcal{G}} - \omega]^{-1} \hat{W}_{\mathcal{G}}(\omega) , \quad (22)$$

and where $g_{\mathcal{G}}(0)$ is the value of $g_{\mathcal{G}}(t)$ at $t=0$ and $\phi_{\mathcal{G}}(\omega)$ can be constructed from the values at $t=0$ of $\phi^{(1)}(t)$ and its time derivatives. The contour C for the Laplace transforms must be above all singularities of $\hat{\phi}^{(1)}(\omega)$ and $\hat{g}_{\mathcal{G}}(\omega)$. The inverse of the operator $[L_{\mathcal{G}} - \omega]$ is nonsingular except on the real ω axis because L is a Hermitian operator.¹ Singularities off the real axis and not associated with initial conditions are singularities of $D^{-1}(\omega)$. The operator $D(\omega)$ is called the dispersion operator. It plays a crucial role in determining the stability properties of the system, and it will be important in our discussion of numerical approximation schemes. Note that the dispersion operator does not depend on the operators $P_{\mathcal{G}}$ in terms of which auxiliary functions $g_{\mathcal{G}}$ may be defined.

It can be shown¹ that the singularities of $\hat{\phi}^{(1)}(\omega)$ and $\hat{g}_{\mathcal{G}}(\omega)$ are located at the roots of the equation

$$\left[\prod_{\mathcal{G}} \det(L_{\mathcal{G}} - \omega \mathbf{I}) \right] \det D(\omega) = 0 , \quad (23)$$

where \mathbf{I} is the unit operator in $(\underline{r}, \underline{v})$ space. Because $L_{\mathcal{G}}$ is Hermitian, any root of this equation which has an imaginary part, corresponding to exponential growth or decay, must be a zero of $\det D(\omega)$. The stability of the system (16)-(17) can be studied numerically by finding a suitable finite-dimensional approximation to $D(\omega)$. A numerical approximation to the solution of the initial-value problem for a fixed length of time can be obtained by finding a suitable approximation of the roots of (23) in terms of a finite number of points in the ω plane. Since the eigenvalue spectrum of $L_{\mathcal{G}}$ is a set of real continua, approximating the roots of (23) with a finite number of points means approximating continuous spectra with discrete spectra of finite size. We now turn to the question of constructing numerical approximations.

III. NUMERICAL ASPECTS OF THE PROBLEM

For numerical purposes it is useful to introduce eigenfunctions of L_a as a basis for the $(\underline{r}, \underline{v})$ space for species a . This is also useful for some analytical calculations. The eigenfunctions of L_a are a complete set of functions in $(\underline{r}, \underline{v})$ space which we take to be orthonormal:¹

$$L_a w_{ar} = \mu_{ar} w_{ar} , \quad (24)$$

$$(w_{ar}, w_{ar'}) = \delta_{rr'} , \quad (25)$$

where the index r stands for whatever set of labels is needed to specify w_{ar} , the parenthesis notation denotes an inner product, and $\delta_{rr'}$ stands for a product of Kronecker deltas and Dirac delta functions--one Kronecker delta for each pair of discrete labels, and one Dirac delta function for each pair of continuous labels. If at most one coordinate is nonignorable in the equilibrium configuration, then the eigenfunctions w_{ar} and eigenvalues μ_{ar} can be found explicitly in terms of definite integrals.¹ The eigenvalues μ_{ar} play an important role in resonance denominators arising from the inverse of the operator $[L - \omega]$ which appears in the Laplace transforms (20)-(21).

In order to have a matrix representation of $D(\omega)$, we also introduce a basis for configuration space with basis functions $\eta_n(\underline{r})$. In some cases, the basic equations can be formulated such that one of the eigenfunctions of the operator Λ which appears in (17) is a good approximation to $\tilde{\phi}^{(1)}(\omega_0)$ for a complex frequency ω_0 of interest. Then, choosing the basis functions η_n to be eigenfunctions of Λ is advantageous,

$$\Lambda \eta_n = \lambda_n \eta_n . \quad (26)$$

The operator Λ can be chosen to be Hermitian for most problems. However, it may not be Hermitian, in which case it can be useful to define a dual set of functions $\zeta_n(\underline{r})$,

$$(\zeta_n, \eta_{n'}) = \delta_{nn'} . \quad (27)$$

where again $\delta_{nn'}$ stands for a product of Kronecker deltas and Dirac delta functions.

Even if one of the eigenfunctions of Λ is not a good approximation to $\hat{\psi}^{(1)}(\omega_0)$, a linear combination of a few of the eigenfunctions may be good; in this case it would still be a good idea to let the basis functions u_n be eigenfunctions of Λ because a small truncated matrix representation of $D(\omega_0)$ could be a good approximation. Of course, the optimal choice of basis would be one with which the representation of $D(\omega)$ would be diagonal; then, $\det D(\omega) = 0$ would be satisfied by setting any diagonal element to zero. If the equilibrium is spatially homogeneous, $D(\omega)$ can be made diagonal by choosing basis functions proportional to $\exp(i\mathbf{k}\cdot\mathbf{r})$, a familiar situation in the stability theory for an infinite homogeneous equilibrium in plasma physics. However, when the equilibrium is spatially inhomogeneous, the basis functions which diagonalize $D(\omega)$ generally will depend on ω , and are usually not known for any given ω .

There is a systematic way of determining a Λ whose eigenfunctions are good basis functions. Sometimes the procedure can be carried out. The idea is to try to diagonalize $D(\omega)$ for a mode of interest whose frequency is near $\omega = \omega_0$. If the basis functions are eigenfunctions of Λ , then the operators \hat{P}_g must satisfy

$$\hat{W}_g(\omega_0) = 0,$$

which means

$$i(L_g - \omega_0)\hat{P}_g(\omega_0)\hat{\psi}^{(1)}(\omega_0) = \hat{U}_g(\omega_0)\hat{\psi}^{(1)}(\omega_0). \quad (28)$$

This corresponds to an approximate solution of the original linearized Boltzmann equation. The parameter ω_0 is determined by solving the dispersion relation obtained from the approximate solution of (28). The \hat{P}_g determined in this way depends parametrically on ω_0 and it determines, through (18), a Λ whose eigenfunctions form a suitable basis for working with a severely truncated dispersion matrix in the neighborhood of $\omega = \omega_0$.

The procedure just outlined for determining the basis functions for configuration space is good if it can be carried out. For example, it was effective in studying the stability of a large-amplitude Bernstein-Greene-Kruskal equilibrium.³ However, it does occur for some problems of physical interest that Λ has to be chosen rather carefully. For example, in the case of a magnetized plasma, there are modes for which spatial variations of short scale length, on the order of the ion gyroradius, are present in addition to variations of much longer scale length.^{4,7} For the procedure of using only a few of the eigenfunctions of Λ to be

effective, they must be able to represent all of the important spatial variations with sufficient accuracy, the short scale length variations as well as those with long scale length. For a specific problem it may not be possible to solve (28) to sufficient accuracy, or it may be inconvenient to do so.

When it is not feasible to use a very small truncation of $D(\omega)$ by taking the eigenfunctions of a suitable Λ as the basis for configuration space, it is necessary to choose a basis with which arbitrary variations of the appropriate scale lengths can be adequately represented. This can easily lead to a matrix representation of $D(\omega)$ whose dimension is large enough that very serious computer storage problems are encountered. If there is more than one nonignorable coordinate in the equilibrium, there is at present no general numerical procedure for finding the eigenfrequencies of the system. Even in the case of one nonignorable coordinate, there has not been a generally applicable numerical procedure so far. Recently, however, an approach has been found which should render it feasible to find the eigenfrequencies numerically for a general equilibrium with one nonignorable coordinate.⁸ The approach makes detailed use of the explicit form of the eigenfunctions and eigenvalues of the operator L_g .¹ It organizes the computational work in a way which minimizes computer storage problems, and relies on some empirical simplicity of the dependence on ω of elements of the dispersion matrix. A computer code based on this approach is being constructed.

An approximation to the solution of the initial-value problem for (15)-(17) can be obtained by approximating the entire spectrum of solutions of (23), including the continuous branches. The continuous part of the spectrum is responsible for phase mixing in the evolution of the system, an example of which is a decay in time of electrostatic perturbations which is known as Landau damping. If a finite set of basis functions w_{gr} and η_n is used, as will be the case in computational work, then (23) is in fact always a polynomial equation.¹ Despite the fact that the polynomial can be of very large degree, it can be evaluated because (23) represents the polynomial as the product of the determinant of a diagonal matrix times the determinant of the dispersion matrix. The determinant of the diagonal matrix is trivial to evaluate, and the determinant of the dispersion matrix can be evaluated if the number of basis functions η_n is not too large. All of the roots of such a polynomial equation of large degree can be found simultaneously by means of a quadratically convergent iteration method proposed by Aberth,⁹ Kerner,¹⁰ and Durand.¹¹ The method also converges when there are multiple roots.

IV. CONCLUSION

The equations governing the small-signal response of spatially inhomogeneous collisionless plasmas have practical significance in physics, for example in controlled thermonuclear fusion research. Although the solutions are very complicated and the equations are difficult to solve numerically, effective methods for them are being developed which are applicable when the equilibrium involves only one nonignorable coordinate. The general theoretical framework probably will provide a basis for progress when there are two or three nonignorable coordinates.

REFERENCES

1. H. R. Lewis and K. R. Symon, *J. Math. Phys.* 20, 413 (1979); also Errata, to be published in *J. Math. Phys.*
2. H. R. Lewis and L. Turner, *Nucl. Fusion* 16, 992 (1976).
3. J. L. Schwarzmeier, H. R. Lewis, B. Abraham-Shrauner, and K. R. Symon, *Phys. Fluids* 22, 1747 (1979).
4. C. E. Seyler, to be published in *Phys. Fluids*.
5. C. E. Seyler and J. P. Freidberg, to be published in *Phys. Fluids*.
6. J. P. Freidberg, *Phys. Fluids* 15, 1102 (1972).
7. T. E. Cayton and H. R. Lewis, to be published in *Phys. Fluids*.
8. C. E. Seyler and H. R. Lewis, work in progress.
9. O. Aberth, *Math. Comp.* 27, 339 (1973).
10. I. O. Kerner, *Numer. Math.* 8, 290 (1966).
11. E. Durand, *Solutions Numériques des Equations Algébriques*, Tome I (Masson, Paris, 1960), pp. 277-281.