

CONF-780831--1

LA-UR -78-757

TITLE: SPHERICAL SHOCK COLLAPSE IN A NON-IDEAL MEDIUM


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SUBMITTED TO: Joint IUTAM/IMU Symposium,
Group Theoretical Methods in Mechanics
Novosibirsk, USSR
August 25-29, 1978

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SPHERICAL SHOCK COLLAPSE IN A NON-IDEAL MEDIUM

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1. Introduction

Scaling laws have always had a place in the study of fluid motion. Fluid motions which are scale-invariant are self-similar motions. The first self-similar motions were studied by Taylor,¹ Sedov,² Guderley,³ and Landau and Stanyukovich.⁴ Self-similar motions in one dimension satisfy ordinary differential equations in scale-invariant variables rather than partial differential equations in space and time. They are physically important because they represent asymptotic states of motion, which occur when the fluid is no longer strongly influenced by its initial conditions.⁵

The scale transformations which determine self-similar motions are Lie group operations. Birkhoff⁶ in 1950 first applied group theory to get invariant solutions of Euler's equations. Subsequently several others -- for example Ovsjannikov,⁷ Michal,⁸ and Müller and Matschat⁹ -- have refined Lie's original method¹⁰ of integration of differential equations by group theoretical techniques. Many have applied it to study group invariant motions of an ideal gas.¹¹

Isentropic fluid motion is governed by Euler's equations. Euler's equations contain three independent dimensions: mass, length, and time. So at most they can admit three independent scale transformations. Self-similar shock motions utilize all three independent choices of scale. One choice is fixed by the initial density ahead of the shock. Another is determined by the numerical solution for the similarity exponent. This also determines the shock trajectory in space-time and the shape of the flow behind the shock. The remaining choice of scale can be used to

determine by a scale transformation the profile of the flow at a later time from its profile at an earlier time. Thus the name self-similar.

Self-similar motions are not admitted for an arbitrary equation of state. Generally the bulk modulus relation, $B_s(p,v)$, which appears in Euler's equations removes some scale freedom by connecting pressure, p , with specific volume, v . Initial and boundary conditions also tend to remove scale freedom by introducing characteristic dimensions.

Non-ideal fluid motions are studied here. Spherical self-similar convergence is calculated for a strong shock in a non-ideal medium. Group theory is used to place a symmetry condition on the adiabatic bulk modulus, $B_s(p,v)$, for which three independent scale transformations of Euler's equations will be admitted. The types of non-ideal media which satisfy the bulk modulus symmetry condition include equations of state of Mie-Grüneisen type. Thus the theory applies to a wide class of materials. In particular it applies to non-degenerate solids at shock pressures well above the yield stress.

In section 2 we state the problem and describe the method of approach. In section 3 we summarize results from invariance analysis of Euler's equations. Then in section 4 we use these results to determine equations of state for which self-similar solutions are admitted. Two such equations of state are those of Tait-Kirkwood-Murnaghan¹¹ and Walsh¹² which have been used before as empirical interpolation functions without the realization that self-similar solutions exist for them.

In section 5 we describe the numerical method we use to solve the self-similar problem for the Walsh equation of state. In section 6 we interpret the numerical solutions and study their stability. It turns out that self-similar shock convergence is stable in radial size but not in sphericity. The same result was obtained by Butler¹³ for an ideal gas, but by another method. Also in section 6 we describe how self-similar analysis may be applied to finite strength shocks.

2. Statement of Problem and Geometric Approach to Its Solution

Consider the problem of the spherical convergence of a shock wave to the center of a uniform, stationary material. Ahead of the shock the initial conditions are

$$p = p_0, \quad \rho = \rho_0, \quad u = 0 \quad (2-1)$$

At the shock front the boundary conditions are given by the Hugoniot jump conditions, with D the shock velocity and E the specific internal energy.

$$\frac{u}{D} = 1 - \frac{v}{v_0}$$

$$p - p_0 = \rho_0 u D \quad (2-2)$$

$$E - E_0 = \frac{1}{2} (p + p_0) (v_0 - v) = \frac{u^2}{2} + p_0 (v_0 - v)$$

For a strong shock the terms in p_0 are to be ignored.

The pressure-volume response of the material is described by the adiabatic bulk modulus,

$$B_s(p, v) = -v \left(\frac{\partial p}{\partial v} \right)_s \quad (2-3)$$

In the absence of viscosity and heat transfer, the motion of the shock front and the flow behind the shock are governed by Euler's equations. These are the continuity equations,

$$\frac{d\rho}{dt} + \rho \left(u_x + \frac{2u}{x} \right) = 0 \quad (2-4)$$

the motion equation,

$$\frac{du}{dt} + \frac{1}{\rho} \Gamma_x = 0 \quad (2-5)$$

and the energy or entropy equation,

$$\frac{dE}{dt} + E_s(p, v) \left(u_x + \frac{2u}{x} \right) = 0 \quad (2-6)$$

where x is the spatial coordinate (radius) and $\frac{d}{dt}$ is the material derivative with respect to time.

The equation of state influences the equations of motion through the adiabatic bulk modulus in the last equation. When written in terms of a general adiabatic bulk modulus, invariance analysis leads to the construction of self-similar solutions and other types of invariant solutions.

for shocks in media other than an ideal gas. We seek functional forms of the adiabatic bulk modulus for which Euler's equations admit the maximal group of point transformations.

3. Invariance Principles for Euler's Equations

Euler's equations (2-4) to (2-6) admit a three parameter subgroup of scale transformations generated by the operator,

$$Q_{op} = (2a_1 + a_3) x \partial_x + (a_1 + a_2) t \partial_t + a_1 u^2 \partial_u + (a_2 - 2a_1) \rho \partial_\rho + a_2 (p + p_e) \partial_p \quad (3-1)$$

provided the adiabatic bulk modulus satisfies the condition,

$$a_2 (p + p_e) \frac{\partial B_s}{\partial p} + (a_2 - 2a_1) \rho \frac{\partial B_s}{\partial \rho} - a_2 B_s = 0 \quad (3-2)$$

where p_e is an arbitrary constant with units of pressure. In the last two equations the arbitrary constants a_1 , a_2 , and a_3 correspond to choices of three direction coefficients in the group space. The general solution of (3-2) for the adiabatic bulk modulus is

$$B_s(p, \rho) = (p + p_e) f \left(\frac{(p + p_e) \rho^{\frac{2a_1}{a_2}}}{c} \right) \quad (3-3)$$

where f is an arbitrary function of its argument. When the bulk modulus has this form, three independent scale transformations are admitted by Euler's equations.

Euler's equations are also invariant under time translations, since the independent variable t does not appear explicitly in the system. So the zero-value of time may be chosen arbitrarily. In planar geometry, invariance under spatial displacements and Galilean transformations would also occur. In more spatial dimensions rigid rotations of all vectors would be admitted, as well.

For the initial condition of uniform density ahead of the shock to be invariant, a relation must be imposed in Q_{op} ,

$$a_2 - 2a_1 = 0 \quad (3-4)$$

Thus the bulk modulus for self-similar shock propagation into a uniform medium adopts the separable form,

$$B_s(p, \rho) = (p + p_e) f(\rho) \quad (3-5)$$

where again $f(\rho)$ is an arbitrary function. Such equations for the bulk modulus have been used before as interpolation functions in shock wave physics. Two choices for $f(\rho)$ are well-known; the TKM equation,¹³

$$f(\rho) = \text{constant} = \frac{1}{A\rho_0} \quad (3-6)$$

and the Walsh equation,¹⁴

$$f(\rho) = \frac{\text{const}}{\rho} = \frac{1}{A\rho} \quad (3-7)$$

The Walsh equation has the added advantage that it is consistent with the experimentally observed linear relation between shock speed, D , and particle speed, u , behind the shock,

$$D = c + su \quad (3-8)$$

true for plate-impact experiments with shock pressure greater than about fifty kilobars. In terms of this relation the constants A, p_e , in the Walsh equation are

$$A = \frac{v_0}{4s} = \frac{1}{\rho_0 4s}, \quad p_e = \frac{\rho_0 c^2}{4s} \quad (3-9)$$

Typically for metals s is about 1.25 and c is roughly equal to the sound speed. In what follows we specialize to the Walsh equation. Comparison with shock wave data of the Mie-Grüneisen equations of state implied by the Walsh and TKM equations for the adiabatic bulk modulus will be published elsewhere.¹⁷

4. Construction of Similarity Variables as Group Invariants

Euler's equations can be reduced to a system of three nonlinear ordinary differential equations by transformation of variables to the invariant coordinates of Q_{op} , the operator in (3-1) with condition (3-4),

$$Q_{op} f(x, t, u, \rho, p) = 0 \quad (4-1)$$

In general the solution of such a first order partial differential equation involves arbitrary functions of the functionally independent integrals of the characteristic equations. In our case the arbitrary

functions are the new dependent variables in Euler's equations, whose solutions are restricted to the invariant surface. Thus the flow variables are resolved from the independent group invariants to be,

$$\lambda = \frac{x}{t^\alpha}, \quad u = \frac{x}{t} U_s(\lambda), \quad \rho = \rho_0 R_s(\lambda), \quad (p + p_e) = \left(\frac{x}{t}\right)^2 \rho_0 p_s(\lambda) \quad (4-2)$$

where the exponent α is,

$$\alpha = \frac{2a_1 + a_3}{a_1 + a_3} \quad (4-3)$$

and the value of time is taken to be negative before collapse and to vanish when the shock reaches the center.

Upon substitution of the self-similar flow variables into Euler's equations, a coupled set of three nonlinear ordinary differential equations in λ remains to be integrated. The boundary and initial conditions for this system will be invariant if the shock trajectory follows a path,

$$x_s(t) = (\text{const}) t^\alpha \quad (4-4)$$

and if also the initial density distribution is uniform,

$$\rho(x, 0) = \rho_0 \quad (4-5)$$

5. Evaluation of the Similarity Variables

The numerical evaluations of the similarity variables requires solving a system of nonlinear ordinary differential equations obtained by entering the group invariants into the conservation relations. This system can be expressed in the following matrix form:

$$\begin{bmatrix} U_s - \alpha & R_s & 0 \\ 0 & U_s - \alpha & R_s^{-1} \\ 0 & \Gamma P_s & (U_s - \alpha) R_s \end{bmatrix} \begin{bmatrix} R'_s \\ U'_s \\ P'_s \end{bmatrix} = \begin{bmatrix} -3U_s R_s \\ U_s - U_s^2 - 2P_s R_s^{-1} \\ -3\Gamma U_s P_s - 2P_s R_s (U_s - 1) \end{bmatrix} \quad (5-1)$$

where $\Gamma = (\rho_0 A)^{-1}$ and primes are derivatives with respect to $\log \lambda$. Solving equation (5-1) explicitly for the derivatives by Cramer's rule produces

$$R'_s = \frac{\Delta_1}{\Delta}, \quad U'_s = \frac{\Delta_2}{\Delta}, \quad P'_s = \frac{\Delta_3}{\Delta} \quad (5-2)$$

in which the determinants are defined as follows:

$$\begin{aligned}
 \Delta &= (U_s - \alpha)R_s \left[(U_s - \alpha)^2 - \Gamma P_s R_s^{-2} \right] \\
 \Delta_1 &= R_s^2 \left\{ 2(1 - \alpha) P_s R_s^{-1} - U_s (U_s - \alpha) [3(U_s - \alpha) + 1 - U_s] \right\} \\
 \Delta_2 &= (U_s - \alpha)R_s \left\{ 3\Gamma U_s P_s R_s^{-2} + U_s (1 - U_s)(U_s - \alpha) - 2(1 - \alpha)P_s R_s^{-1} \right\} \\
 \Delta_3 &= (U_s - \alpha)P_s \left\{ 2\Gamma P_s R_s^{-1} + 2R_s (1 - U_s)(U_s - \alpha) - \Gamma U_s [3(U_s - \alpha) + 1 - U_s] \right\}.
 \end{aligned} \tag{5-3}$$

Division among equations (5-2) gives,

$$\frac{dR_s}{dU_s} = \frac{\Delta_1}{\Delta_2} \cdot \frac{dP_s}{dU_s} = \frac{\Delta_3}{\Delta_2} \tag{5-4}$$

The similarity exponent α which appears in equations (5-4) can not be determined from an integral energy balance as is the case for diverging shock waves driven by the release of energy at a point. The procedure for finding the numerical values of the similarity exponent for various values of Γ entails solving equations (5-4) numerically, and iterating on assumed values for the similarity exponent. In this procedure the Rankine-Hugoniot relations, which for this problem are

$$\begin{aligned}
 U_s(\lambda_H) &= \frac{2\alpha}{\Gamma} - \frac{2}{\alpha} \left(\frac{P_0 + B}{\rho_0} \cdot \frac{t^{2(1-\alpha)}}{\lambda_H^2} \right) \\
 P_s(\lambda_H) &= \frac{2\alpha^2}{\Gamma} - \left(\frac{P_0 + B}{\rho_0} \cdot \frac{t^{2(1-\alpha)}}{\lambda_H^2} \right) \\
 R_s(\lambda_H) &= \left(1 - \frac{2}{\Gamma} + \frac{2}{\alpha^2} \frac{P_0 + B}{\rho_0} \cdot \frac{t^{2(1-\alpha)}}{\lambda_H^2} \right)^{-1}
 \end{aligned} \tag{5-5}$$

must be satisfied. In the strong shock limit these general forms of the Rankine-Hugoniot relations simplify to

$$U_s(\lambda_H) = \frac{2\alpha}{\Gamma} \quad , \quad P_s(\lambda_H) = \frac{2\alpha^2}{\Gamma} \quad , \quad R_s(\lambda_H) = \frac{\Gamma}{\Gamma - 2} \tag{5-6}$$

When the correct value of the similarity exponent has been chosen, the numerical integration of (5-4) subject to the initial conditions (5-6)

results in single-valued nonsingular functions. The test for convergence resides in the fact that only two of the four determinants in equations (5-3) are linearly independent. When any two of these four determinants vanish simultaneously at the same value of α , the correct value for the similarity exponent has been selected.

6. Numerical Results, Stability, and Finite-Strength Shocks

The solution to equations (5-1) can be visualized as sketched in Figure 1 in the space with coordinates (t, x, u) . In Figure 1 group transformations generated by Q_{op} produce a vector field tangential to the solution surface along lines $\lambda = \text{const}$. The group-reduced Euler equations produce a tangential vector field which crosses lines $\lambda = \text{const}$. More quantitatively, for $\Gamma = 5$ numerical results are graphed in Figure 2 for the reduced variables $U_s(\lambda)$, $P_s(\lambda)$, and $R_s(\lambda)$. The functions $U_s(\lambda)$ and $P_s(\lambda)$ are decreasing functions, while $R_s(\lambda)$ is an increasing function just behind the shock position at $\lambda = 1$.

In Figure 3 is shown the dependence of the similarity exponent, α , on the material parameter $\Gamma = \frac{1}{\gamma - 1}$, with specific materials labeled. Superimposed as a dotted line in Figure 3 is the similarity exponent, α , as a function of $\gamma = c_p/c_v$ for an ideal gas. The two curves disagree by about ten percent near the middle of the range shown.

The stability of these solutions has also been examined. The major conclusion is that an angular perturbation will grow and oscillate as the shock converges, as though the shock had surface tension. The perturbation is stable in size but not in shape as the shock converges to the origin. The physical interpretation of this stability behavior: near the origin the shock becomes non-radially coupled to itself and ceases to converge to a single point. Details of the stability analysis will be published elsewhere.¹⁰

The invariance analysis presented here may be extended to finite shock strengths essentially without change, except for a parametric dependence of the similarity exponent on the value of the time before collapse. The group analysis and numerical procedure described here still apply for finite-strength shock collapse, but the numerical iteration for the similarity exponent must be done as a function of time. Details for finite-strength shocks will be published elsewhere.¹⁰

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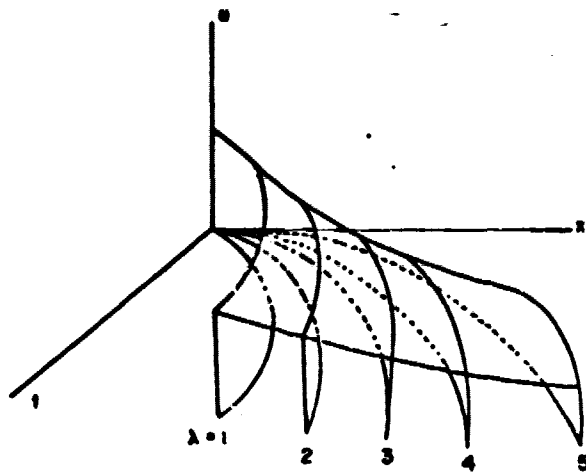


Fig. 1. The solution surface in the space (t,x,u).

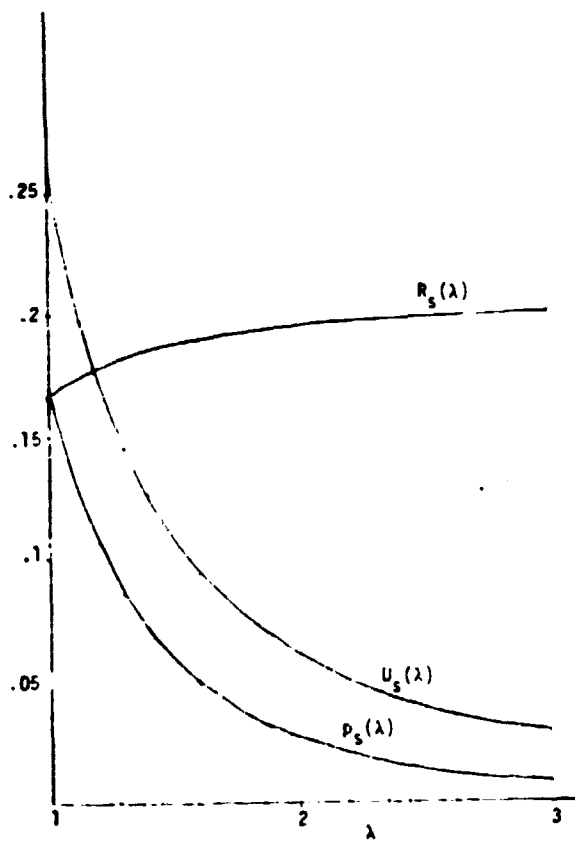


Fig. 2. The group-reduced dependent variables as functions of the similarity variable for $\Gamma=5$, $\alpha=0.6512t$.

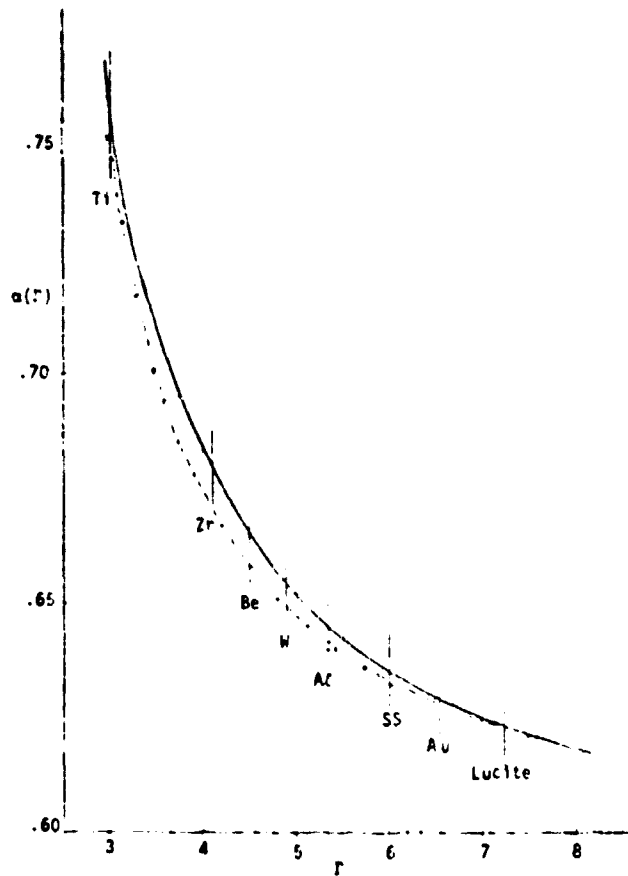


Fig. 3. The dependence of the similarity exponent α on the material parameter

$$\Gamma = \frac{1}{A\rho_0} = 45 .$$