Proof that Nonlinear Plane Waves cannot be destabilized by Scalar Diffusion

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We begin with the reaction–diffusion equation:

$$\frac{\partial u}{\partial t} = \mathbf{D}\frac{\partial^2}{\partial x^2}u + \mathbf{D}\nabla_\perp^2 u + F(u) \qquad 1$$

where u is a *vector* of chemical concentrations and $\mathbf{D}$ is the diffusion matrix. Here $\nabla_\perp^2$ refers to directions orthogonal to x. Assume that there exists a plane wave solution to Eq. (1). Denote this solution by U(x-ct). Define $\xi = x - ct$. Then we have

$$-cU_\xi = \mathbf{D}U_{\xi\xi} + F(U) \quad . \qquad 2$$

Now we ask under what conditions is the plane wave solution stable in multiple dimensions. To do this we work in the travelling coordinate system and write:

$$u(\xi, y, t) = U(\xi) + \eta(\xi, y, t) \qquad 3$$

where η is a small perturbation to the plane wave solution. We expand η in Fourier modes in y. (The dimensionality of the system is irrelevant. For simplicity I work only in 2 space dimensions here, ξ and y.) Thus we have:

$$\eta = \sum_k a_k(\xi)\phi_k(y)\exp(\lambda_k t) \qquad 4$$

where the ϕ_k are eigenfunctions of $\nabla_\perp^2$. Inserting equations (3) and (4) into (1) we find the eigenvalue problem for λ_k:

$$\lambda_k a_k(\xi) = \mathbf{D}\frac{\partial^2 a_k}{\partial \xi^2} - \mathbf{D}k^2 a_k(\xi) + J(\xi)a_k(\xi) + c\frac{\partial a_k}{\partial \xi} \qquad 5$$

In the case that $\mathbf{D} = D\mathbf{I}$ we have

$$\lambda_k = \lambda_0 - Dk^2 \quad . \qquad 6$$

Equation (6) means that diffusion stabilizes small perturbations of the wave front in the case of scalar diffusion which is the desired result.