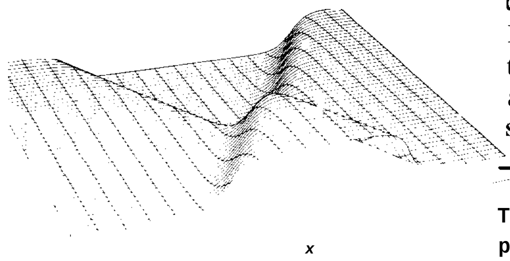
SOLITONS in the SINE-GORDON Equation



INTERACTION BETWEEN TWO SOLITONS

To understand quantitatively how solitons can result from a delicate balance of dispersion and nonlinearity, let us begin with the *linear, dispersionless,* bidirectional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0. \qquad (1)$$

By direct substitution into Eq. 1, it is easy to check that, with $\eta \equiv x - c_0 t$ and $\xi \equiv x + c_0 t$,

$$u(x,t) = f(\eta) + g(\xi)$$

is a solution for *any* functions f and g. Thus if we take, for example, $f(\eta) = e^{-\eta^2}$ and $g(\xi) = e^{-\xi^2}$, we will have two "solitary waves," one moving to the left and one to the right. At $t \to -\infty$, the lumps are isolated, at $t = O$ they collide, and at $t \to +\infty$ they re-emerge. Hence, by our definition these solutions to the linear, dispersionless wave equation are solitons, although trivial examples of such.

The robustness of solitons is indicated in this projected space-time plot of the interaction of a kink and an antikink in the sine-Gordon equation (Eq. 5). As time develops (toward the reader), the two steplike solitary waves approach each other, interact nonlinearly, and then emerge unchanged in shape, amplitude, and velocity. The collision process is described analytically by Eq. 9. (The figure was made at the Los Alamos National Laboratory by Michel Peyrard, University of Bourgogne, France.)

Now consider an equation, still linear, of the form

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + m^2 u = 0. \qquad (2)$$

Such equations arise naturally in descriptions of optically-active phonons in solid state physics and in relativistic field theories. An elementary (plane wave) solution of this equation has the form

$$u(x,t) = Ae^{i(\omega t + kx)}, \qquad (3)$$

where A is a constant, w is the frequency, and k is the wave number. Substituting into Eq. 2 shows that this plane wave can be a solution of Eq. 2 *only if*

$$-\omega^2 + c_0^2 k^2 + m^2 = 0, \qquad (4a)$$

so that

$$\omega(k) = \pm\sqrt{m^2 + c_0^2 k^2}. \qquad (4b)$$

This relation between w and k is known technically as a *dispersion relation* and contains essential information about how individual plane waves with different k's (and hence different w's) propagate. In particular, the group velocity,

$$v_g(k_0) \equiv \frac{\partial \omega}{\partial k}\Big|_{k_0},$$

measures how fast a small group of waves with values of k peaked around a particular value k_0 moves. Notice that for the dispersion relation Eq. 4b,

$$v_g(k_0) = \pm \frac{c_0^2 k_0}{\sqrt{c_0^2 k_0^2 + m^2}},$$

corresponding variety of real-world applications to problems in the natural sciences and engineering. In fiber optics, conducting polymers and other quasi-one-dimensional systems, Josephson transmission lines, and plasma cavitons—as well as the surface waves in the Andaman Sea!—the prevailing mathematical models are slight modifications of soliton equations. There now exist several numerical and analytic perturbation techniques for studying these "nearly" soliton equations, and one can use these to describe quite accurately the behavior of real physical systems.

One specific, decidedly practical illustration of the application of solitons concerns effective long-distance communication by means of optical fibers. Low-intensity light pulses in optical fibers propagate linearly but dispersively (as described in "Solitons in the Sine-Gordon Equation"). This dispersion tends to degrade the signal, and, as a consequence, expensive "repeaters" must be added to the fiber at regular intervals to reconstruct the pulse.

However, if the intensity of the light transmitted through the fiber is substantially increased, the propagation becomes nonlinear and solitary wave pulses are formed. In fact, these solitary waves are very well described by the solitons of the "nonlinear Schrodinger equation," another of the celebrated completely integrable nonlinear partial differential equations. In terms of the (complex) electric field amplitude $E(x, t)$, this equation can be written so that (groups of) waves with different values of k_0 will have different group velocities. Now consider a general solution to Eq. 2, which, by the principle of superposition, can be formed by adding together many plane waves (each with a different constant). Since the elementary components with different wave numbers will propagate at different group velocities, the general solution will change its form, or disperse, as it moves. Hence, the general solution to Eq. 2 cannot be a soliton.

Next consider adding a nonlinear term to Eq. 2. With considerable malice aforethought, we change notation for the dependent variable and choose the nonlinearity so that the full equation becomes

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2} + m^2\sin\theta = 0, \qquad (5)$$

the "sine-Gordon" equation. We can compare Eq. 5 to our previous Eq. 2 by noting that in the limit of small θ, Eq. 5 reduces to

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2} + m^2\theta - \frac{1}{6}\theta^3 + \cdots = 0, \quad (6)$$

where the remaining terms are $\mathcal{O}(\theta^5)$ and higher.

Based on remarks made in the introductory section of the main text, we see that Eq. 5 looks like a bunch of simple, plane pendulums coupled together by the spatial derivative term $\partial^2\theta/\partial x^2$. In fact, the sine-Gordon equation has many physical applications, including descriptions of chain-like magnetic compounds and transmission lines made out of arrays of Josephson junctions of superconductors. Also, the equation is one of the celebrated completely integrable, infinite-degree-of-freedom Hamiltonian systems, and the initial-value problem for the equation can be solved exactly by the analytic technique of the "inverse spectral transform." Since the details of this method are well beyond the scope of a general overview, we shall only quote the solutions relevant to our discussion. First, just as for the KdV equation (Eq. 10 in the main text), one can find directly a single solitary-wave solution:

$$\theta_k(x,t) = 4\tan^{-1} e^{\gamma(\zeta - v\tau)}, \qquad (7)$$

with $\gamma = 1/\sqrt{1-v^2}$, $\zeta = mx/c_0$, and $\tau = mt$.

Since this solution approaches 0 as $x \to -\infty$ and 2π as $x \to +\infty$, it describes a glitch in the field θ localized around $\zeta = v\tau$, that is, around $x = c_o vt$. As a consequence, it is known as a "kink." Importantly, it does represent a physically truly localized excitation, because all the energy and momentum associated with this wave are exponentially centered around the kink's location. Similarly, the so-called anti-kink solution

$$\theta_{\bar{k}}(x,t) = 4\tan^{-1} e^{-\gamma(\zeta - v\tau)}, \qquad (8)$$

interpolates between 2π as $x \to -\infty$ and 0 as $x \to +\infty$.

Are the kinks and anti-kinks solitons? Here we can avail ourselves of the miracle of integrability and simply write down an analytic solution that describes the scattering of a kink and an antikink. The result is

$$\theta_{k\bar{k}}(x,t) = 4\tan^{-1}\left(\frac{\sinh\frac{v\tau}{\sqrt{1-v^2}}}{v\cosh\frac{\zeta}{\sqrt{1-v^2}}}\right). \qquad (9)$$

The dedicated reader can verify that as $t \to -\infty$, $\theta_{k\bar{k}}$ looks like a widely separated kink and anti-kink approaching each other at velocity v. For t near 0 they interact nonlinearly, but as $t \to +\infty$, the kink and anti-kink emerge with their forms intact. Readers with less dedication can simply refer to the figure, in which the entire collision process is presented in a space-time plot. Note that since the equation is invariant under $\theta \to \theta + 2n\pi$, a kink that interpolates between 2π and 4π is physically equivalent to one that interpolates between 0 and 2π.

In the interest of historical accuracy, we should add one final point. The analytic solution, Eq. 9, showing that the kink and anti-kink are in fact solitons, was actually known, albeit not widely, *before* the discovery of the KdV soliton. It had remained an isolated and arcane curiosity, independently rediscovered several times but without widespread impact. That such solutions could be constructed analytically in a wide range of theories was not appreciated. It took the experimental mathematics of Zabusky and Kruskal to lead to the soliton revolution. ■

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + |E|^2E = 0. \qquad (12)$$

The soliton corresponding to the nonlinear pulse moving with velocity v through the optical fiber has the form

$$E(x,t) = \left(2\omega + \frac{v^2}{2}\right)^{\frac{1}{2}} e^{i\omega t + \frac{vx}{2}} \operatorname{sech}\left(\left(\omega + \frac{v^2}{4}\right)^{\frac{1}{2}}(x - vt)\right). \qquad (13)$$

In the idealized limit of no dissipative energy loss, these solitons propagate without degradation of shape; they are indeed the natural stable, localized modes for propagation in the fiber. An intrinsically nonlinear characteristic of this soliton, shown explicitly in Eq. 13, is the relation between its amplitude (hence its energy) and its width. In real fibers, where dissipative mechanisms cause solitons to lose energy, the individual soliton pulses therefore broaden (but do not disperse). Thus, to maintain the separation between solitons necessary for the integrity of the signal, one must add optical amplifiers. based on stimulated Raman amplification, to compensate for the loss.

Theoretical numerical studies suggest that the amplification can be done very effectively. An all-optical system with amplifier spacings of 30 to 50 kilometers and