The Simple but NONLINEAR PENDULUM

Elementary physics texts typically treat the simple plane pendulum by solving the equation of motion only in the linear approximation and then presenting the general solution as a superposition of sines and cosines (as in Eq. 3 of the main text). However, the full nonlinear equation can also be solved analytically in closed form, and a brief discussion of this solution allows us to illustrate explicitly several aspects of nonlinear systems.

It is most instructive to start our analysis using the Hamiltonian for the simple pendulum, which, in terms of the angle θ (a generalized coordinate) and the corresponding (generalized) momentum $p_\theta \equiv ml^2\frac{d\theta}{dt}$, has the form

$$H(p_\theta,\theta) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta. \qquad (1)$$

Using the Hamiltonian equations

$$\frac{\partial H}{\partial p_\theta} = \frac{d\theta}{dt},$$

$$\text{and} \quad \frac{\partial H}{\partial \theta} = -\frac{dp_\theta}{dt},$$

we obtain (after substituting for p_θ) an equation solely in terms of the angle θ and its derivative:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \qquad (2)$$

Recognizing that $\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = \frac{d}{dt}(\frac{1}{2}(\frac{d\theta}{dt})^2)$ and $\frac{d\theta}{dt}\sin\theta = \frac{d}{dt}(-\cos\theta)$, we see that Eq. 2 can be converted to a perfect differential by multiplying by $d\theta/dt$:

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta\frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{l}\cos\theta\right) = 0. \qquad (3)$$

Hence, we can integrate Eq. 3 immediately to obtain

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{l}\cos\theta + C. \qquad (4)$$

By comparing Eqs. 1 and 3 and recalling the definition of p_θ, we see that

$$C = \frac{H(p_\theta,\theta)}{ml^2}.$$

That the constant C is proportional to the *value* of the Hamiltonian, of course, is just an expression of the familiar conservation of energy and shows that the value of the conserved energy determines the nature of the pendulum's motion.

Restricting our considerations to librations—that is, motions in which the pendulum oscillates back and forth without swinging over the top of its pivot point—we can evaluate C in terms of θ by using the condition $d\theta/dt = 0$ when $\theta = \theta_{max}$, which yields

$$C = -\frac{g}{l}\cos\theta_{max}.$$

This, in turn, means that

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_{max})}. \qquad (5)$$

The full period of the motion T is then the definite integral

$$T = 4\int_0^{\theta} \frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_{max})}}. \qquad (6)$$

This last integral can be converted, via trigonometric identities and redefinitions of variables, to an elliptic integral of the first kind. Although not as familiar as the sines and cosines that arise in the linear approximation, the elliptic integral is tabulated and can be readily evaluated. Thus, the full equation of motion for the nonlinear pendulum can be solved in closed form for arbitrary initial conditions.

An elegant method for depicting the solutions for the one-degree-of-freedom system is the "phase plane." If we examine such a plot (see Fig. 2 in the main text), we see that the origin ($\theta = 0$, $d\theta/dt = 0$)—and, of course, its periodic equivalents at $\theta = \pm 2n\pi$, $d\theta/dt = 0$—represent *stable fixed points* with the pendulum at rest and the bob pointing down. The point at $\theta = \pi$, $d\theta/dt = 0$—and, again, its periodic equivalents at $\theta = \pm(2n+1)\pi$, $d\theta/dt = 0$—represent *unstable fixed points* with the pendulum at rest but the bob inverted; the slightest perturbation causes the pendulum to move away from these points. The closed curves near the horizontal axis ($d\theta/dt = 0$) represent librations, or *periodic oscillations*. The open, "wavy" lines away from the horizontal axis (large $|d\theta/dt|$) correspond to *unbounded motions* in the sense that 6’ increases or decreases forever as the pendulum rotates around its pivot point in either a clockwise ($d\theta/dt < 0$) or a counterclockwise ($d\theta/dt > 0$) sense.

What about other systems? A dynamical system that can be described by *2N* generalized position and momentum coordinates is said to have N degrees of freedom. Hamiltonian systems that, like the pendulum, have only one degree of freedom can *always* be integrated completely with the techniques used for Eqs. 2-6. More generally, however, systems with N degrees of freedom are *not* completely integrable; Hamiltonian systems with N degrees of freedom that *are* completely integrable form a very restricted but extremely important subset of all N-degree-of-freedom systems.

As suggested by the one-degree-of-freedom case, complete integrability of a system with N degrees of freedom requires that the system have N constants of motion—that is, N integrals analogous to Eq. *4-and* that these constants be consistent with each other. Technically, this last condition is equivalent to saying that when the constants, or integrals of motion, are expressed in terms of the dynamical variables (as C is in Eq. 4), the expressions must be "in involution," meaning that the Poisson brackets must vanish identically for all possible pairs of integrals of motion. Remarkably, one can find nontrivial examples of completely integrable systems, not only for N-degree-of-freedom systems but also for the "infinite’ ’-degree-of-freedom systems described by partial differential equations. The sine-Gordon equation, discussed extensively in the main text, is a famous example.

In spite of any nonlinearities, systems that are completely integrable possess remarkable regularity, exhibiting smooth motion in all regions of phase space. This fact is in stark contrast to nonintegrable systems. With as few as one-and-a-half degrees of freedom (such as the damped, driven system with three generalized coordinates represented by Eq. 4 in the main text), a nonintegrable system can exhibit deterministic chaos and motion as random as a coin toss. ■