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A NOTE ON THE CALCULATION OF NEUTRON MULTIPLICATION

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PHYSICS AND MATHEMATICS

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A NOTE ON THE CALCULATION OF NEUTRON MULTIPLICATION

1. Definition of Neutron Multiplication.

We consider a spherical system with outer radius a cm comprised of a central core and M-1 shells arranged in a concentric fashion. With this system we associate a steady neutron source S which we assume to be locally isotropic as well as spherically symmetric in distribution. We denote the source S thus specified by S(r), $0 \leq r \leq a$, and assign to it the dimension neuts/cm^{3}sec. The multiplication M of the sphere is then defined as the number of neutrons eventually emerging from the sphere per neutron emitted by the source S. M is clearly a function of the source S as well as the properties of the spherical system.

2. Problems Considered in this Report.

The numerical methods based on integral theory and introduced in LA-1271 are applicable under very general assumptions to the problem of calculating M. The purpose of this report is to present a simple semi-analytical procedure applicable to one-medium spherical systems under the assumptions of one-velocity isotropic theory. This means that we limit ourselves to a small but important class of problems involving monoenergetic sources. We shall measure r in units of the mean free path $1/\sigma$, and denote σr and c-1 by x and f, respectively. The cases we consider are then fully described by x, f, and S. For the definitions of σ and c and the assumptions underlying one-velocity

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isotropic theory we refer to LA-1271.

We shall also restrict the discussion to the particular source distributions given below. These are of considerable theoretical as well as practical importance.

(1) Central point source S_c of strength S_o neuts/sec, with

$$S_c(r) = S_o\, \delta(r) \text{ and } T_c = \int_o^a S_c(r) r^2 dr = S_o \text{ neuts/sec.}$$

(2) Uniform source S_u of strength S_o neuts/cm^{3}sec, with

$$S_u(r) = S_o \text{ and } T_u = \frac{4\pi}{3} a^3 S_o \ .$$

(3) Surface source S_s of uniform strength S_o neuts/cm^{2}sec on the surface of the sphere with $S_s(r) = S_o\, \delta(a-r)$ and

$$T_s = 4\pi a^2 S_o \ .$$

3. Approximate Formulae for M.*

In the simple cases we consider here, it is possible to give an approximate but nevertheless quite accurate formula for M, the error being in most cases of the order of a few per mil. From the definition of M we have immediately:

(4) $$M = (1-P_1) + (1+f)P_1(1-P_2) + (1+f)^2 P_1 P_2 (1-P_2) + \ldots\ldots ,$$

* For other reports on the calculation of neutron multiplication see references on Page 10.

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where P_k, k=1,2,3,..., is defined as the probability that a source neutron, after k-1 previous collisions in the sphere, will have at least a k^{th} collision. P_k evidently depends on x and S. Moreover, P_k approaches P_n as k goes to infinity, where P_n depends on x alone. This follows from integral theory considerations. A so-called normal mode distribution is established characteristic of the size of the sphere. Once a collection of neutrons are distributed in the normal mode they remain so distributed though their number may change.

The approximate formula for M is obtained from the assumption that the distribution of first collisions can be written as a combination of the source distribution and the normal mode. This gives rise to the following formula for P_1P_2:

(5) $$P_1P_2 = \left[\beta P_1\right] P_1 + \left[(1-\beta)P_1\right] P_n ,$$

where β will be determined by a procedure to be described later.

The above assumption implies that the distribution of k^{th} collisions likewise can be written as a combination of the source distribution and the normal mode. This in turn leads to formula (6) below for the probability $\mathcal{P}_k$ that a source neutrons will have at least k collisions in the sphere:

(6) $$\mathcal{P}_k = \prod_{\ell=1}^{k} P_\ell = \left[\beta \mathcal{P}_{k-1}\right] P_1 + \left[(1-\beta)P_1P_n^{k-2}\right] P_n, \quad k=2,3,4,\ldots$$

An interpretation of the two brackets in (6) can be given. The first represents the fraction of the source neutrons which after k-1 collisions

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are given the distribution of the source. The second represents the corresponding fraction given the normal mode distribution.

From (6) we deduce the following simple recursion formula for P_k:

$$P_k = P_n + \beta P_1 (P_{k-1} - P_n)/P_{k-1} \quad . \tag{7}$$

The approximate formula for M can now be derived. As a first step we rearrange the terms in (4) obtaining:

$$M = 1 + f \left[\mathcal{P}_1 + (1+f)\mathcal{P}_2 + (1+f)^2 \mathcal{P}_3 + \ldots . \right] \quad . \tag{8}$$

The sum of the terms in the bracket of (8) are then found analytically with the aid of (6). The resulting expression for M is given by:

$$M = 1 + \frac{P_1}{P_n} \frac{1-\beta(1+f)P_n}{1-\beta(1+f)P_1} \frac{fP_n}{1-(1+f)P_n} \quad , \tag{9}$$

where P_1 and β depend on x and S, and P_n on x. In particular, if S is a so-called normal mode source, i.e., a source having the same distribution as the normal mode, we find that $P_1 = P_n$, and $\beta = 1$. Hence:

$$M_n = 1 + \frac{fP_n}{1-(1+f)P_n} = \frac{1}{1 - \frac{P_n}{1-P_n} f} \quad . \tag{10}$$

4. <u>Calculation Procedure for P_1, P_n, and β.</u>

The values of P_1 were obtained from the following analytic formulae:

$$P_{c1} = 1 - e^{-x} \quad , \tag{11}$$

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(12) $$P_{ul} = 1 - \frac{3}{8x^3}\left[(2x^2-1)+(2x+1)e^{-2x}\right],$$

(13) $$P_{sl} = \frac{1}{2} - \frac{1}{4x}\,(1-e^{-2x}) \ .$$

The successive values of P_k and hence P_n were, on the other hand, obtained numerically using the integral theory methods of LA-1271, splitting the interval (o,x) in four equal parts. The distribution of the neutrons emerging from first collisions could, however, be derived analytically for each of the three sources under consideration. For the sake of greater accuracy these distributions were used in the integral equation in place of the corresponding source distributions. It would have been quite inaccurate to use the latter especially in the case of the δ -sources S_c and S_s which for large x would be spread considerably in a four-interval scheme.

The formulae for the distributions $S_1(t,x)$ of first collisions are given below:

(14) $$S_{c1}(t) = xe^{-tx}\ , \qquad 0 \leq t \leq 1\ ,$$

(15) $$S_{ul} = 3t^2\left\{1 - \frac{1}{2tx}\left[E_3(x(1-t))-E_3(x(1+t))\right] - \frac{1}{2t}\left[E_2(x(1-t))-E_2(x(1+t))\right]\right\},$$

(16) $$S_{sl} = \frac{tx}{2}\left[E_1(x(1-t))-E_1(x(1+t))\right]\ ,$$

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where $tx = \sigma r$, $0 \leq r \leq \sigma a = x$, and $\int_0^1 S_1(t,x)dt = P_1$.

The values of β were calculated from the following formula:

(17) $$\beta = \frac{P_n K + 1}{P_1 K - 1} ,$$

where K is defined as the limit as k approaches infinity of $\mathcal{P}_k/P_n^k$.

5. Approximate Formulae for $\left[d(1/M)/dm\right]$ and $\left[d\alpha/dm\right]$.

Other quantities of particular interest in connection with critical mass determinations are $\left[d(1/M)/dm\right]_{m=m_o}$ and $\left[d\alpha/dm\right]_{m=m_o}$, where m_o is the critical mass in kg and α the exponential growth (or decay) factor in reciprocal shakes. The system becomes critical when M approaches infinity which occurs when x goes to x_o, x_o being the root of the equation $(1+f)P_n(x) = 1$. Consequently, we have $m_o = \frac{4\pi}{3000}\rho\left(\frac{x_o}{\sigma}\right)^3$, where ρ is the density in gr/cm^3. From the formula for M given above and the procedure for time-dependent problems (See LA-1271, Section VI) we obtain:

(18) $$\left[\frac{d(1/M)}{dx}\right]_{x=x_o} = -\frac{1}{fx_oP_1}\,\frac{1-(1+f)\beta P_1}{1-\beta}\,\frac{x_oP'_n}{P_n} ,$$

(19) $$\left[\frac{d\alpha}{dx}\right]_{x=x_o} = \frac{\sigma v}{x_o}\,\frac{x_oP'_n/P_n}{1-(x_oP'_n/P_n)} ,$$

(20) $$\left[\frac{dx}{dm}\right]_{m=m_o} = \frac{1000\sigma^3}{4\pi\rho x_o^2} ,$$

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where v in (19) is the average neutron velocity in cm/shake.

The average number of collisions $\bar{w}$ generated in the sphere per source neutron emitted is also of some interest. Since each collision gives rise to f new neutrons we may write M in terms of $\bar{w}$, $M=1+\bar{w}f$, from which we obtain:

$$(21) \qquad \bar{w} = (M-1)f$$

The number of fissions in the system is then given by $\frac{\sigma f}{\sigma t}\,\bar{w} = (M-1)/(\nu-1)$.

6. Numerical Examples.

(a) Find the critical mass of an oralloy sphere with $\sigma = .28$, $f = .30$, and $\rho = 18.8$. From Table I we find, corresponding to $P_n(x_o) = 1/(1+f) = .7692$, the value 2.435 for x_o. Using the formula for m_o given above we obtain $m_o = 51.79$ kg.

(b) Find $\left[\frac{d(1/M_c)}{dm}\right]_{m=m_o}$ and $\left[\frac{d\alpha}{dm}\right]_{m=m_o}$ for an oralloy sphere of mass 51.79 kg. From Table I we find $P_1 = .9124$, $P_n(x_o) = .7692$, $\beta(x_o) = .6387$ and $x_oP'_n(x_o)/P_n(x_o) = .337$. Substituting these numbers in formulae (18), (19), and (20), letting v = 9.5 cm/shake, we obtain $\left[d(1/M)/dm\right]_{m=m_o} = .00532$, and $\left[d\alpha/dm\right]_{m=m_o} = .0087$.

(c) Find M_c, M_u, and M_s, and M_n for an oralloy sphere with $\sigma = .28$, $f = .30$, $\rho = 18.8$, and $a = 7.143$. We than have $x = 2.000$ and $m = 28.70$. With the aid of Tables I, II, and III we find the

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following values of P_1, P_n, and β :

Source	P_1	P_n	β
M_c	.8647	.7155	.5946
M_u	.6676	.7155	.4654
M_s	.3773	.7155	.4093
M_n	.7155	.7155	1.0000

Substituting these in the formula (10) for M we obtain $M_c = 6.01$, $M_u = 3.73$, $M_s = 2.26$, and $M_n = 4.07$.

Selected References

LA-191, J. H. Manley and R. L. Walker, "Multiplication of a 2-1/2 Inch 25 Sphere as Measured with 28 and 25 Fission Detectors".

LA-235, W. Rarita and R. Serber, "Critical Masses and Multiplication Rates".

LA-267, C. Richman, "Multiplication of Neutrons in Small Spheres of Active Material".

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LAMS-227, A. O. Hanson, R. Serber, and J. H. Williams, "Multiplication by Small Spheres of Active Material".

LAMS-230, A. O. Hanson, R. Serber,and J. H. Williams, "Multiplication of Large 25 Spheres".

LAMS-235, A. O. Hanson, R. Serber, and J. H. Williams, "Multiplication of Spheres of 49".

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TABLE I CENTRAL SOURCE DATA

x	$P_1(x)$	$P_2(x)$**	$P_n(x)$	$K(x)$	$\beta(x)$	xP_n'/P_n
0	1.00000x*	.8651x*	.7828x*	1.5136	.3600	1.000
.4	.32968	.2893	.2584	1.5763	.4084	.820
.8	.55067	.4918	.4352	1.6292	.4571	.676
1.2	.69881	.6356	.5596	1.6729	.5047	.563
1.6	.79810	.7387	.6493	1.7099	.5509	.472
2.0	.86466	.8129	.7155	1.7405	.5946	.400
2.4	.90928	.8664	.7654	1.7661	.6353	.342
2.8	.93919	.9048	.8038	1.7872	.6728	.294
3.2	.95924	.9325	.8338	1.8053	.7069	.256
3.6	.97268	.9522	.8577	1.8201	.7376	.224
4.0	.98168	.9663	.8769	1.8318	.7650	.198
4.4	.98772	.9763	.8927	1.8422	.7895	.176
4.8	.99177	.9834	.9057	1.8507	.8111	.157

* For small values of x. ** From four-interval integral theory calculations.

Interpolation: Find K and β by quadratic interpolation, P_1 from $P_1 = 1 - e^{-x}$, and P_n from $P_n = P_1\left[(1-\beta) + \beta K\right]/K$. Quadratic interpolation in $\log(1-P_k)$, $k = 1, 2, \ldots n$, is also good.

TABLE II UNIFORM SOURCE DATA

x	$P_1(x)$	$P_2(x)^{**}$	$P_n(x)$	K(x)	$\beta(x)$
0	.75000x	.7770x	.7828x	.9468	.2213
.4	.24536	.2555	.2584	.9318	.2761
.8	.41045	.4289	.4352	.9176	.3280
1.2	.52508	.5498	.5596	.9045	.3774
1.6	.60713	.6363	.6493	.8925	.4234
2.0	.66758	.6997	.7155	.8816	.4654
2.4	.71333	.7475	.7654	.8718	.5032
2.8	.74881	.7843	.8038	.8631	.5367
3.2	.77693	.8133	.8338	.8555	.5664
3.6	.79966	.8365	.8577	.8489	.5924
4.0	.81834	.8555	.8769	.8431	.6153
4.4	.83394	.8713	.8927	.8380	.6357
4.8	.84714	.8846	.9057	.8335	.6542

* For small values of x. ** From four-interval integral theory calculations.

Interpolation: Find K and β by quadratic interpolation, P_1 by linear interpolation in Table IV, and P_n from $P_n = P_1 \left[(1-\beta) + \beta K\right]/K$. Quadratic interpolation in $\log(1-P_k)$, $k = 1, 2, \ldots n$, is also good.

TABLE III SURFACE SOURCE DATA

x	$P_1(x)$	$P_2(x)^{**}$	$P_n(x)$	$K(x)$	$\beta(x)$
0	.50000x	.7399x	.7828x	.5841	.2056
.4	.15583	.2405	.2584	.5325	.2504
.8	.25059	.3998	.4352	.4893	.2941
1.2	.31057	.5084	.5596	.4531	.3356
1.6	.35012	.5853	.6493	.4228	.3741
2.0	.37729	.6418	.7155	.3972	.4093
2.4	.39669	.6850	.7654	.3756	.4409
2.8	.41104	.7192	.8038	.3571	.4692
3.2	.42200	.7471	.8338	.3412	.4945
3.6	.43061	.7706	.8577	.3275	.5171
4.0	.43752	.7908	.8769	.3155	.5370
4.4	.44319	.8084	.8927	.3053	.5543
4.8	.44792	.8240	.9057	.2967	.5688

* For small values of x. ** From four-interval integral theory calculations.

Interpolation: Find K and β by quadratic interpolation,

P_1 by linear interpolation in Table IV,

and P_n from $P_n = P_1 \left[(1-\beta) + \beta K \right] / K$.

Quadratic interpolation in log $(1-P_k)$,

$k = 1, 2, \ldots n$, is also good.

x	Central Source	Uniform Source	Surface Source
.00	.00000	.00000	.00000
.05	.04877	.03652	.02418
.10	.09516	.07116	.04683
.15	.13929	.10403	.06803
.20	.18127	.13525	.08790
.25	.22120	.16490	.10653
.30	.25918	.19308	.12401
.35	.29531	.21987	.14042
.40	.32968	.24536	.15583
.45	.36237	.26962	.17032
.50	.39347	.29273	.18394
.55	.42305	.31473	.19676
.60	.45119	.33573	.20883
.65	.47795	.35573	.22020
.70	.50341	.37481	.23093
.75	.52763	.39304	.24104
.80	.55067	.41045	.25059
.85	.57259	.42708	.25961
.90	.59343	.44298	.26814
.95	.61326	.45819	.27620
1.00	.63212	.47275	.28383
1.05	.65006	.48668	.29106
1.10	.66713	.50003	.29791
1.15	.68336	.51282	.30440
1.20	.69881	.52508	.31057
1.25	.71350	.53684	.31642
1.30	.72747	.54812	.32198
1.35	.74076	.55896	.32726
1.40	.75340	.56937	.33229
1.45	.76543	.57937	.33707
1.50	.77687	.58898	.34163
1.55	.78775	.59823	.34598
1.60	.79810	.60713	.35012
1.65	.80795	.61569	.35407
1.70	.81732	.62394	.35785
1.75	.82623	.63189	.36146
1.80	.83470	.63955	.36491
1.85	.84276	.64694	.36821
1.90	.85043	.65407	.37136
1.95	.85773	.66094	.37439
2.00	.86466	.66758	.37729
2.05	.87127	.67400	.38007
2.10	.87754	.68019	.38274
2.15	.88352	.68618	.38530
2.20	.88920	.69197	.38776
2.25	.89460	.69758	.39012
2.30	.89974	.70300	.39240
2.35	.90463	.70825	.39458
2.40	.90928	.71333	.39669
2.45	.91371	.71826	.39872
2.50	.91792	.72303	.40067

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TABLE IV, PROBABILITY OF FIRST COLLISION (P_1). (Cont.)

x	Central Source	Uniform Source	Surface Source
2.50	.91792	.72303	.40067
2.55	.92192	.72766	.40256
2.60	.92573	.73214	.40438
2.65	.92935	.73650	.40613
2.70	.93279	.74072	.40783
2.75	.93607	.74483	.40946
2.80	.93919	.74881	.41104
2.85	.94216	.75268	.41257
2.90	.94498	.75644	.41405
2.95	.94766	.76009	.41549
3.00	.95021	.76365	.41687
3.05	.95264	.76711	.41822
3.10	.95495	.77047	.41952
3.15	.95715	.77374	.42078
3.20	.95924	.77693	.42200
3.25	.96123	.78003	.42319
3.30	.96312	.78305	.42435
3.35	.96492	.78600	.42546
3.40	.96663	.78887	.42655
3.45	.96825	.79167	.42761
3.50	.96980	.79440	.42864
3.55	.97128	.79706	.42964
3.60	.97268	.79966	.43061
3.65	.97401	.80219	.43155
3.70	.97528	.80466	.43247
3.75	.97648	.80708	.43337
3.80	.97763	.80944	.43424
3.85	.97872	.81174	.43509
3.90	.97976	.81399	.43592
3.95	.98075	.81619	.43673
4.00	.98168	.81834	.43752
4.05	.98258	.82044	.43829
4.10	.98343	.82250	.43904
4.15	.98424	.82451	.43977
4.20	.98500	.82648	.44049
4.25	.98574	.82841	.44119
4.30	.98643	.83029	.44187
4.35	.98709	.83213	.44254
4.40	.98772	.83394	.44319
4.45	.98832	.83571	.44383
4.50	.98889	.83744	.44445
4.55	.98943	.83914	.44506
4.60	.98995	.84081	.44566
4.65	.99044	.84244	.44624
4.70	.99090	.84403	.44681
4.75	.99135	.84560	.44737
4.80	.99177	.84714	.44792
4.85	.99217	.84865	.44846
4.90	.99255	.85012	.44898
4.95	.99292	.85157	.44950
5.00	.99326	.85300	.45000

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