

6

Radiation and Radiative Transfer

In the preceding chapters we treated the physics of nonradiating fluids; we now extend the analysis to *radiating fluids* comprising both material and radiation. Radiation adds to the total energy density, momentum density, stress, and energy flux in the fluid. We must therefore define these quantities for radiation and derive equations that describe the coupling among them and their coupling to the material.

Our first goal is to develop an understanding of the “microphysics” of the radiation field and of transport processes in the combined matter-radiation fluid along lines conceptually similar to our study of gases in Chapters 1 and 3. For this purpose it suffices to assume that the material is static, which is what we generally do in this chapter. Detailed discussion of how radiation transports energy and momentum through moving media, and couples to the dynamics of flows, is reserved for Chapter 7.

In §6.1 we derive expressions that specify the basic dynamical properties of the radiation field, in particular its energy density, energy flux, and stress tensor; we specialize these to the case of thermal equilibrium in §6.2. We then turn to the principal task of this chapter: the formulation and solution of the *transfer equation*, which determines how radiation is transported through the material. In §6.3 we describe the interaction of radiation with material in terms of macroscopic absorption and emission coefficients. Then in §6.4 we derive the transfer equation, which is the equivalent of the Boltzmann equation for photons (cf. §92), and discuss the significance of its moments.

In §6.5 we discuss methods for solving the transfer equation. In opaque material, such as the interior of a star, photons are trapped and the radiation field is nearly isotropic and approaches thermal equilibrium; the photon mean free path λ_p is much smaller than a characteristic structural length l in the material. In this limit, radiative energy transport can be described as a diffusion process, and we can derive an asymptotic solution of the transfer equation, which is similar to the Chapman–Enskog solution of the Boltzmann equation describing transport phenomena in gases (see also §97). Like the Chapman–Enskog solution, radiation diffusion theory is valid only for $\lambda_p \ll l$.

But as we approach a boundary surface of a radiating medium (e.g., the atmosphere of a star), the material becomes transparent, and photon mean

free paths can vastly exceed characteristic structural lengths. Here a *nonlocal* treatment is needed. We must solve the full transfer equation describing photon exchange within the fluid, thereby in effect constructing a nonlocal kinetic theory for photons. The nonlocal nature of the transfer problem is exacerbated by the *scattering* of photons by matter. After being thermally emitted, a photon may scatter many times, changing essentially only its direction of travel, before being absorbed and destroyed by reconversion into thermal energy. In doing so, a photon migrates a large distance called the *photon destruction length*. The radiation field is then no longer uniquely determined by local conditions but at any point may be determined by conditions within a large *interaction volume* whose size is set by a photon destruction length, not a mean free path. Hence *the radiation field is not, in general, a local variable*.

Because the transfer equation is (superficially) linear in the radiation field, it is possible to solve it for very general physical situations by powerful numerical techniques. Actually the transfer problem is linear only to the extent that we consider the material absorption and emission coefficients as *given*. But in reality these coefficients depend on the internal excitation and ionization state of the material, and, as we discuss in §6.6, this state is fixed in part by radiative processes that populate and depopulate atomic levels. We therefore find that in general *the radiation field and the internal state of the matter must be determined simultaneously and self-consistently*.

In the diffusion regime, the radiation field and level populations have their *thermal equilibrium* distributions and the coupling between radiation and matter presents no difficulty. Somewhat nearer to a radiating surface we reach a regime in which significant nonlocal radiation transport occurs, but collisional processes still dominate the state of the material, which can be calculated from the equations of statistical mechanics evaluated at local values of the temperature and density—the *local thermodynamic equilibrium* (LTE) regime. When the medium is very transparent, and photons escape freely from a boundary surface into space, the radiation field takes on a strongly *nonequilibrium* character. We must then reconsider the microphysics of the gas, allowing for a nonequilibrium interaction between the radiation and material: this poses a difficult problem both mathematically and conceptually because the local state of the material is then coupled by photon exchange to the state of the material within an entire interaction volume. The techniques required to solve this interlocked problem are discussed in §6.7.

This chapter forms essential background for the discussion of radiation hydrodynamics in Chapter 7. Conceptually the goal of these two chapters is to develop formalisms that describe accurately the strong interactions between radiation and matter in radiating fluids. It will pay the reader to reread this chapter after reading Chapter 7, having the benefits of insights gained there. Although it is our intent to give a self-contained account of

the topics treated in this chapter, some of the material is technical, and we must sometimes omit details. We recommend that the reader consult references such as (A2), (A3), (J1), (K1), (M2), (P1), (P3), (S1), (T2), (U1) and (W1) for further background and amplification.

6.1 The Radiation Field

63. The Specific Intensity and Photon Distribution Function

The radiation field is, in general, a function of position and time, and at any given position has a distribution in both angle and frequency. We define the *specific intensity* $I(\mathbf{x}, t; \mathbf{n}, \nu)$ of radiation at position \mathbf{x} and time t , traveling in direction \mathbf{n} with frequency ν , to be such that the amount of energy transported by radiation of frequencies $(\nu, \nu + d\nu)$ across a surface element dS , in a time dt , into a solid angle $d\omega$ around \mathbf{n} , is

$$d\mathcal{E} = I(\mathbf{x}, t; \mathbf{n}, \nu) dS \cos \alpha d\omega d\nu dt, \quad (63.1)$$

where α is the angle between \mathbf{n} and the normal to dS . In cgs units, I has dimensions $\text{ergs cm}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{sr}^{-1}$.

In most of what follows, we confine attention to one-dimensional structures and flows in planar or spherical geometry. In the planar case, we assume that the material is homogeneous in the horizontal direction, with properties varying only as a function of z and t . The intensity then has azimuthal symmetry around the unit vector \mathbf{k} ; its angular distribution can be described completely in terms of the polar angle Θ or $\mu \equiv \cos \Theta = \mathbf{n} \cdot \mathbf{k}$. Hence $I = I(z, t; \mu, \nu)$. We assume that z is positive in the direction opposite to gravity, and explicit mention of z and t will normally be suppressed.

In spherical geometry, a position is specified by (r, θ, ϕ) , and the direction of radiation at that position by polar and azimuthal angles (Θ, Φ) measured with respect to the radial unit vector $\hat{\mathbf{r}}$. For *spherical symmetry* I depends on r only, and is independent of Φ ; therefore $I = I(r, t; \mu, \nu)$ where now $\mu = \mathbf{n} \cdot \hat{\mathbf{r}}$. Explicit mention of r and t will usually be suppressed.

The specific intensity provides a complete *macroscopic* description of the radiation field. From a *microscopic* view, the radiation field is composed of photons, and we define the *photon number density* ψ such that $\psi(\mathbf{x}, t; \mathbf{n}, \nu) d\omega d\nu$ is the number of photons per unit volume at (\mathbf{x}, t) with frequencies $(\nu, \nu + d\nu)$, traveling with velocity c into a solid angle $d\omega$ around \mathbf{n} . The number of photons crossing a surface element $d\mathbf{S}$ in time dt is then $\psi(\mathbf{n} \cdot d\mathbf{S})(d\omega d\nu)(c dt)$. Each photon has energy $h\nu$, so the energy transported is

$$d\mathcal{E} = ch\nu\psi dS \cos \alpha d\omega d\nu dt. \quad (63.2)$$

Comparing (63.2) with (63.1) we find

$$I(\mathbf{x}, t; \mathbf{n}, \nu) = ch\nu\psi(\mathbf{x}, t; \mathbf{n}, \nu). \quad (63.3)$$

A relative of ψ is the *photon distribution function* f_R , defined such that $f_R(\mathbf{x}, t; \mathbf{n}, p) d^3p$ is the number of photons per unit volume at (\mathbf{x}, t) with momenta $(\mathbf{p}, \mathbf{p} + d\mathbf{p})$, where $\mathbf{p} = (h\nu/c)\mathbf{n}$. Using $d^3p = p^2 dp d\omega = (h/c)^3 \nu^2 d\nu d\omega$, we find $(h^3 \nu^2 / c^3) f_R d\nu d\omega \equiv \psi d\nu d\omega$, and therefore

$$I(\mathbf{x}, t; \mathbf{n}, \nu) = (h^4 \nu^3 / c^2) f_R(\mathbf{x}, t; \mathbf{n}, \nu). \quad (63.4)$$

The function f_R is a relativistically invariant distribution function (cf. §43) describing certain massless, extreme-relativistic particles (photons) in a six-dimensional phase space; it is completely analogous to the invariant particle distribution function used in §43 to construct a kinetic theory for a relativistic gas. In §§90 and 91 we use a kinetic theory approach to develop expressions for the radiative energy density, energy flux, and stress in terms of f_R and its moments. Furthermore, in §§92 and 95 we describe the interaction between radiation and matter by a Boltzmann equation for f_R . But for the present, we emphasize the continuum view, and most of the analysis in this chapter is done in terms of I .

64. The Mean Intensity and Radiation Energy Density

The *mean intensity* J is defined as the average of the specific intensity over all solid angles, that is,

$$J_\nu = J(\mathbf{x}, t; \nu) \equiv (4\pi)^{-1} \oint I(\mathbf{x}, t; \mathbf{n}, \nu) d\omega. \quad (64.1)$$

J_ν has dimensions $\text{ergs cm}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{sr}^{-1}$. The mean intensity is the *zeroth moment* of the radiation field over angles.

In a planar atmosphere I is independent of Φ . Thus, noting that $d\omega = \sin \Theta d\Theta d\Phi = -d\mu d\Phi$, we then have

$$J(z, t; \nu) = (4\pi)^{-1} \int_0^{2\pi} d\Phi \int_{-1}^1 d\mu I(z, t; \mu, \nu) = \frac{1}{2} \int_{-1}^1 I(z, t; \mu, \nu) d\mu. \quad (64.2)$$

This result also holds in spherical symmetry with z replaced by r .

The *monochromatic radiation energy density* at frequency ν is the number density of photons at that frequency, summed over all solid angles, times their energy $h\nu$. That is,

$$E_\nu = E(\mathbf{x}, t; \nu) = h\nu \oint \psi(\mathbf{x}, t; \mathbf{n}, \nu) d\omega. \quad (64.3)$$

Using (63.3) and (64.1) we see that

$$E_\nu = c^{-1} \oint I(\mathbf{x}, t; \mathbf{n}, \nu) d\omega = (4\pi/c) J_\nu. \quad (64.4)$$

E_ν has dimensions $\text{ergs cm}^{-3} \text{Hz}^{-1}$. The *total radiation energy density* is

$$E = E(\mathbf{x}, t) \equiv \int_0^\infty E(\mathbf{x}, t; \nu) d\nu = (4\pi/c) \int_0^\infty J(\mathbf{x}, t; \nu) d\nu = (4\pi/c) J(\mathbf{x}, t), \quad (64.5)$$

which has dimensions ergs cm^{-3} .

65. *The Radiative Energy Flux and Momentum Density*

We define the *monochromatic radiation flux* $\mathbf{F}(\mathbf{x}, t; \nu)$ to be a vector such that $\mathbf{F} \cdot d\mathbf{S}$ gives the net rate of radiant energy flow per unit frequency interval, at frequency ν , across $d\mathbf{S}$. The net number flux of photons crossing $d\mathbf{S}$ per unit time and frequency interval from all solid angles is

$$N = \left(\oint \psi(\mathbf{x}, t; \mathbf{n}, \nu) c \mathbf{n} d\omega \right) \cdot d\mathbf{S}, \quad (65.1)$$

which, multiplied by the energy per photon, $h\nu$, gives the energy flux. Recalling that $I = ch\nu\psi$ we see that

$$\mathbf{F}_\nu = \mathbf{F}(\mathbf{x}, t; \nu) = \oint I(\mathbf{x}, t; \mathbf{n}, \nu) \mathbf{n} d\omega, \quad (65.2)$$

or, in components

$$\mathbf{F}_\nu = (F_x, F_y, F_z)_\nu = \left(\oint I_\nu n_x d\omega, \oint I_\nu n_y d\omega, \oint I_\nu n_z d\omega \right), \quad (65.3)$$

where $n_x = (1 - \mu^2)^{1/2} \cos \Phi$, $n_y = (1 - \mu^2)^{1/2} \sin \Phi$, and $n_z = \mu$. In cgs units F_ν has dimensions $\text{ergs cm}^{-2} \text{s}^{-1} \text{Hz}^{-1}$. The flux is the *first moment* of the radiation field over angle. Summing over all frequencies we obtain the *integrated radiation flux*

$$\mathbf{F} = \mathbf{F}(\mathbf{x}, t) \equiv \int_0^\infty \mathbf{F}(\mathbf{x}, t; \nu) d\nu \quad (65.4)$$

which has dimensions $\text{ergs cm}^{-2} \text{s}^{-1}$.

For azimuthal symmetry around \mathbf{k} , F_x and F_y are identically zero; the remaining component F_z is therefore often called “the” flux

$$F_\nu = F(z, t; \nu) \equiv 2\pi \int_{-1}^1 I(z, t; \mu, \nu) \mu d\mu, \quad (65.5)$$

as if it were a scalar. Following Eddington it is customary to define

$$H_\nu = H(z, t; \nu) \equiv (4\pi)^{-1} F(z, t; \nu) = \frac{1}{2} \int_{-1}^1 I(z, t; \mu, \nu) \mu d\mu, \quad (65.6)$$

which is similar to (64.1) for J_ν . Equations (65.5) and (65.6) also apply in spherical symmetry with z replaced by r .

The momentum of a photon with energy $h\nu$ is $(h\nu/c)\mathbf{n}$; therefore the net rate of radiative momentum transport across $d\mathbf{S}$ at frequency ν is $c^{-1}\mathbf{F}_\nu \cdot d\mathbf{S}$. This transport is effected by particles moving with a speed c , hence the *monochromatic radiation momentum density* vector is

$$\mathcal{G}_\nu = c^{-2}\mathbf{F}_\nu. \quad (65.7)$$

Integrating over all frequencies we see that the *total radiation momentum density* is

$$\mathcal{G} = c^{-2}\mathbf{F} \quad (65.8)$$

where \mathbf{F} is the total radiation flux. This result also follows from considerations of the form of the radiation stress-energy tensor (cf. §91).

66. The Radiation Pressure Tensor

As for material particles, we define the *radiation stress tensor*, or *pressure tensor*, \mathbf{P} such that P^{ij} is the net rate of transport, per unit area of a surface oriented perpendicular to the j th coordinate axis, of the i th component of momentum. The number of photons of frequency ν , moving in direction n^j , crossing a unit area in a unit time, is $\psi_\nu cn^j$; each has momentum $(h\nu n^i/c)$ in the i th direction. Thus summing over all solid angles we obtain the *monochromatic radiation pressure tensor*

$$P^{ij}(\mathbf{x}, t; \nu) = \oint \psi(\mathbf{x}, t; \mathbf{n}, \nu) (h\nu n^i/c) (cn^j) d\omega, \quad (66.1)$$

or

$$P^{ij}(\mathbf{x}, t; \nu) = c^{-1} \oint I(\mathbf{x}, t; \mathbf{n}, \nu) n^i n^j d\omega. \quad (66.2)$$

In dyadic notation

$$\mathbf{P}_\nu = \mathbf{P}(\mathbf{x}, t; \nu) = c^{-1} \oint I(\mathbf{x}, t; \mathbf{n}, \nu) \mathbf{nn} d\omega. \quad (66.3)$$

\mathbf{P}_ν is manifestly symmetric, and is clearly the *second moment* of the radiation field over angle; the components of \mathbf{P}_ν have dimensions dynes $\text{cm}^{-2} \text{Hz}^{-1}$.

The rate of momentum transport across an oriented surface element $d\mathbf{S}$ with normal \mathbf{l} , by photons of frequency ν , is $P_\nu^{ij} l_j d\mathbf{S}$. Integrating over a closed surface S surrounding a volume V , we find that the total rate of flow of the i th component of radiation momentum at frequency ν out of V is

$$\int_S P_\nu^{ij} l_j d\mathbf{S} = \int_V (P_\nu^{ij})_{,j} dV. \quad (66.4)$$

In the absence of momentum exchange between radiation and matter (and of body forces that affect photons, that is, general relativity effects), this flow decreases the momentum density \mathcal{G}_ν in V . We therefore must have

$$(\partial \mathcal{G}_\nu / \partial t) = c^{-2} (\partial \mathbf{F}_\nu / \partial t) = -\nabla \cdot \mathbf{P}_\nu, \quad (66.5)$$

which is identical to the momentum equation (23.5) for an ideal fluid in the absence of body forces. We emphasize that (66.5) applies only in the absence of material; interactions with matter are treated in §§78, 93, and 96.

Because I_ν is independent of Φ in a one-dimensional medium, direct calculation from (66.2) yields

$$\mathbf{P}_\nu = \begin{pmatrix} P_\nu & 0 & 0 \\ 0 & P_\nu & 0 \\ 0 & 0 & P_\nu \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3P_\nu - E_\nu & 0 & 0 \\ 0 & 3P_\nu - E_\nu & 0 \\ 0 & 0 & 3P_\nu - E_\nu \end{pmatrix}. \quad (66.6)$$

Here the scalar P_ν is defined as

$$P_\nu \equiv (4\pi/c)K_\nu, \quad (66.7)$$

where, in turn,

$$K_\nu \equiv \frac{1}{2} \int_{-1}^1 I(z, t; \mu, \nu) \mu^2 d\mu. \quad (66.8)$$

In planar geometry P_{11} , P_{22} , and P_{33} in (66.6) represent P_{xx} , P_{yy} , and P_{zz} ; in spherical geometry they represent the physical components $P_{\theta\theta}$, $P_{\phi\phi}$, and P_{rr} relative to the orthonormal triad $(\hat{\theta}, \hat{\phi}, \hat{r})$. Equations (66.6) show that in the special case of a one-dimensional medium two scalars, P_ν and E_ν , suffice to specify the full tensor \mathbf{P}_ν . Moreover, derivatives with respect to (x, y) or (θ, ϕ) must be identically zero by symmetry. Therefore, in planar geometry the only nonvanishing component of $\nabla \cdot \mathbf{P}_\nu$ is

$$(\nabla \cdot \mathbf{P}_\nu)_z = \partial P_\nu / \partial z, \quad (66.9)$$

and, from (A3.91), the only nonvanishing component in spherical symmetry is

$$(\nabla \cdot \mathbf{P}_\nu)_r = (\partial P_\nu / \partial r) + (3P_\nu - E_\nu)/r. \quad (66.10)$$

From (66.9) and (66.10) one can understand why in one-dimensional problems it is customary to refer to the scalar P_ν as “the” radiation pressure. But it is important to bear in mind that because the second term in (66.6) is not necessarily zero, P_ν is not, in general, isotropic, and therefore does not reduce to a simple hydrostatic pressure. The anisotropy of P_ν reflects an anisotropic distribution of $I(\mu, \nu)$, which is induced by efficient photon exchange between regions with significantly different physical properties, particularly in the presence of strong gradients, and/or an open boundary.

Because the trace P_ν^{ii} of \mathbf{P}_ν is an invariant, it is sometimes used to define a *mean radiation pressure*

$$\bar{P}_\nu = \bar{P}(\mathbf{x}, t; \nu) \equiv \frac{1}{3} P_\nu^{ii}(\mathbf{x}, t; \nu) = \frac{1}{3} E(\mathbf{x}, t; \nu), \quad (66.11)$$

the last equality following directly from (66.6). While (66.11) is true in general, note that \bar{P}_ν does *not*, in general, equal P_ν , nor does it have any particular dynamical significance.

The radiation pressure tensor will be isotropic for any distribution of the radiation field $I(\mu, \nu)$ that yields $P_\nu = \frac{1}{3} E_\nu$. A particular example is *isotropic radiation*, for which $K_\nu = \frac{1}{3} J_\nu$ from (64.2) and (66.8), hence $P_\nu = \frac{1}{3} E_\nu$ from (64.4) and (66.7). In this case

$$\mathbf{P}(\mathbf{x}, t; \nu) = \begin{pmatrix} P_\nu & 0 & 0 \\ 0 & P_\nu & 0 \\ 0 & 0 & P_\nu \end{pmatrix} \quad (66.12)$$

and, for computational purposes, the entire radiation pressure tensor can be

replaced by a scalar hydrostatic pressure $P_\nu = \frac{1}{3}E_\nu$. This is a case of great practical importance because it holds true in the diffusion regime (cf. §80), which shows why one can always use scalar pressures in stellar interior calculations. Note also from (66.10) that in this case $(\nabla \cdot \mathbf{P})_r$ reduces to just $(\partial P/\partial r)$. The small departures of \mathbf{P}_ν from isotropy in the diffusion limit in moving media are discussed in §97.

On the other hand, at the boundary of a medium with a positive temperature gradient inward, the radiation field is peaked (i.e., is largest) in the direction of outward flow (cf. §§79 and 82). Moreover, from (66.7) and (64.4) we see that such radiation, with $\mu \approx 1$, is more heavily weighted in \mathbf{P}_ν than in E_ν . Hence near boundary surfaces the ratio P_ν/E_ν usually exceeds $\frac{1}{3}$. The extreme example is a *plane wave* traveling along the z axis, for which $I(\mu) = I_0 \delta(\mu - 1)$. In this case, called the *streaming limit*, $J_\nu = H_\nu = K_\nu$, hence $P_\nu = E_\nu$, and \mathbf{P}_ν has only one nonzero element, namely P_{zz} .

From the discussion above, we see that it is useful to define the dimensionless ratio

$$f_\nu = f(\mathbf{x}, t; \nu) \equiv P(\mathbf{x}, t; \nu)/E(\mathbf{x}, t; \nu) = K_\nu/J_\nu, \quad (66.13)$$

which is known as the *variable Eddington factor*, to give a measure of the degree of anisotropy of the radiation field. For an opaque medium with a boundary, f_ν typically lies in the range $\frac{1}{3}$ to 1. We will see in §§78 and 83 that f_ν can be used to close the system of moments of the transfer equation; this important idea was first suggested by B. E. Freeman (**F2**), and cast into an easily applied form by G. R. Spillman (**S4**).

All quantities defined in this section have frequency-integrated counterparts. For example, the scalar describing the total radiation pressure in a one-dimensional medium is

$$P = P(\mathbf{x}, t) = \int_0^\infty P(\mathbf{x}, t; \nu) d\nu = (4\pi/c) \int_0^\infty K(\mathbf{x}, t; \nu) d\nu, \quad (66.14)$$

which is also the total hydrostatic pressure for isotropic radiation; it has dimensions dynes cm^{-2} . Similarly, the *total radiation pressure tensor* is

$$\mathbf{P} = \mathbf{P}(\mathbf{x}, t) = c^{-1} \int_0^\infty d\nu \oint d\omega I(\mathbf{x}, t; \mathbf{n}, \nu) \mathbf{nn}. \quad (66.15)$$

6.2 Thermal Radiation

In the important limiting case of thermodynamic equilibrium, the radiation field is described by a unique distribution function that depends on only one state variable, the absolute temperature T . Such *thermal radiation* exists in a *hohlraum*, an isolated enclosure in thermal equilibrium at a uniform temperature; in the laboratory, close approximations to a hohlraum are provided by carefully insulated ovens. Another close approximation is the deep interior of a star. For example, in the Sun the mean

temperature gradient from center to surface is about $10^7 \text{ K}/10^{11} \text{ cm} = 10^{-4} \text{ K cm}^{-1}$, while photon mean free paths are a fraction of a centimeter; the radiation field is thus in equilibrium with material at a very uniform temperature, and approaches perfect thermal radiation very closely.

67. Planck's Law

Thermal radiation is described by the *Planck function* $B_\nu(T)$. The functional form of $B_\nu(T)$ follows immediately from Bose–Einstein quantum statistics [see, e.g., (C5, Chap. 10) or (H2, Chap. 12)]; but for our purposes a brief semiclassical derivation suffices [cf. (S2, Chap. 6)].

Planck hypothesized that radiation comes in discrete quanta of energy $h\nu$, and that the radiation field in a hohlraum is a superposition of quantized oscillations or *modes*. In a rectangular cavity of dimensions (X, Y, Z) each mode is characterized by positive integers (n_x, n_y, n_z) such that the propagation vector \mathbf{k} has components $(n_x\pi/X, n_y\pi/Y, n_z\pi/Z)$, which guarantees that the modes are standing waves. For each \mathbf{k} there are two senses of polarization of the field, with electric vectors orthogonal to each other and to \mathbf{k} , defining two modes.

Let us count the number of modes with frequencies on the range $(\nu, \nu + d\nu)$. The magnitude of \mathbf{k} is $k = 2\pi/\Lambda = 2\pi\nu/c$; hence

$$\nu = (ck/2\pi) = \frac{1}{2}c[(n_x/X)^2 + (n_y/Y)^2 + (n_z/Z)^2]^{1/2}. \quad (67.1)$$

Surfaces of constant frequency are thus the ellipsoids

$$(n_x/a_x)^2 + (n_y/a_y)^2 + (n_z/a_z)^2 = 1, \quad (67.2)$$

where $(a_x, a_y, a_z) = (2\nu/c)(X, Y, Z)$. The number of normal modes with frequencies $\nu' \leq \nu$ equals twice (for two polarizations) the number of points with integer coordinates within one octant (all n 's ≥ 0) of the ellipsoid (67.2). The volume of an ellipsoid is $(4\pi a_x a_y a_z/3)$, hence

$$N(\nu) = 2 \times \frac{1}{8} \times (4\pi/3)(8XYZ\nu^3/c^3) = (8\pi\nu^3/3c^3)V, \quad (67.3)$$

where $V = XYZ$ is the volume of the hohlraum. Therefore the number of modes with frequencies on $(\nu, \nu + d\nu)$ is

$$dN = (8\pi\nu^2/c^3)V d\nu. \quad (67.4)$$

To calculate the average energy associated with these modes we assume, with Planck, that the energy of n active modes of frequency ν is $nh\nu$. In equilibrium at temperature T , the relative probability of a set of modes having total energy ε_n is

$$\pi_n = \exp(-\varepsilon_n/kT) / \sum_n \exp(-\varepsilon_n/kT), \quad (67.5)$$

whence the average energy of all modes at frequency ν is

$$\langle \varepsilon \rangle = \left[\sum_{n=0}^{\infty} nh\nu \exp(-nh\nu/kT) \right] / \left[\sum_{n=0}^{\infty} \exp(-nh\nu/kT) \right]. \quad (67.6)$$

Writing $x \equiv \exp(-h\nu/kT)$ we can rewrite the denominator of (67.6) as

$$1 + x + x^2 + \dots = (1 - x)^{-1} = (1 - e^{-h\nu/kT})^{-1}, \quad (67.7)$$

while the numerator is

$$\begin{aligned} h\nu(x + 2x^2 + 3x^3 + \dots) &= h\nu x \frac{d}{dx} (1 + x + x^2 + \dots) \\ &= h\nu x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{h\nu x}{(1-x)^2}. \end{aligned} \quad (67.8)$$

Hence

$$\langle \varepsilon \rangle = h\nu x / (1 - x) = h\nu / (e^{h\nu/kT} - 1). \quad (67.9)$$

The energy density in the hohlraum is the number of modes per unit volume times the average energy per mode. Hence from (67.4) and (67.9) we have

$$E_v^* = E^*(\nu, T) = (8\pi h\nu^3/c^3) / (e^{h\nu/kT} - 1), \quad (67.10)$$

where the asterisk indicates thermal equilibrium. From (64.4) we then see that $B_\nu(T)$, the (isotropic) specific intensity in thermal equilibrium, is

$$B_\nu(T) = (2h\nu^3/c^2) / (e^{h\nu/kT} - 1); \quad (67.11)$$

this distribution characterizes the radiation, usually called *blackbody radiation*, emitted by a *perfect radiator* or *black body*.

68. Stefan's Law

The integrated energy density for thermal radiation is

$$E^*(T) = \int_0^\infty E^*(\nu, T) d\nu = (8\pi h/c^3) \int_0^\infty \nu^3 (e^{h\nu/kT} - 1)^{-1} d\nu. \quad (68.1)$$

Writing $x \equiv h\nu/kT$ we have

$$E^*(T) = (8\pi k^4 T^4 / c^3 h^3) \int_0^\infty x^3 e^{-x} (1 + e^{-x} + e^{-2x} + \dots) dx. \quad (68.2)$$

Integrating the series term by term we find

$$6(1 + 2^{-4} + 3^{-4} + \dots) = 6\zeta_4 = \pi^4/15, \quad (68.3)$$

where ζ_4 is the Riemann zeta function of order four (**A1**, 807). We thus obtain *Stefan's law*, which states that in thermal equilibrium the total radiation energy density is proportional to the fourth power of the absolute temperature, or

$$E^*(T) = a_R T^4, \quad (68.4)$$

where

$$a_R \equiv 8\pi^5 k^4 / 15c^3 h^3. \quad (68.5)$$

From (68.4) and (64.5) one sees that the integrated Planck function is

$$B(T) = (a_R c / 4\pi) T^4. \quad (68.6)$$

It is customary to define the *Stefan-Boltzmann constant* σ_R such that

$$\pi B(T) = \sigma_R T^4, \quad (68.7)$$

whence $\sigma_R = \frac{1}{4} a_R c$. The rationale for this definition follows from calculating the radiation flux emergent from a black body, namely

$$F_{\text{BB}}(\nu) = 2\pi \int_0^1 B_\nu(T) \mu \, d\mu = \pi B_\nu(T), \quad (68.8)$$

which yields an integrated flux

$$F_{\text{BB}} = \pi B(T) = \sigma_R T^4. \quad (68.9)$$

Note that (68.8) and (68.9) apply to radiation *emerging* from a hohlraum; the isotropy of equilibrium radiation implies that the net flux *within* the hohlraum is identically zero.

69. Thermodynamics of Equilibrium Radiation

The radiation field within an equilibrium cavity has associated with it both an energy density and a stress. Energy can be fed into or withdrawn from the cavity, and the radiation field can do mechanical work. In §§69 to 71 we examine the thermodynamic properties of equilibrium radiation, both by itself and accompanied by material. Inasmuch as radiation has no mass, it is awkward to work with intensive variables defined per unit mass; we will therefore use extensive variables. Conversion of our results to quantities per unit volume or mass is straightforward.

Because thermal radiation is isotropic, the monochromatic thermal radiation pressure is, from (66.12)

$$P_\nu^* = \frac{1}{3} E_\nu^* = (4\pi/3c) B_\nu(T), \quad (69.1)$$

whence the total thermal radiation pressure is

$$P^* = \frac{1}{3} E^* = \frac{1}{3} a_R T^4. \quad (69.2)$$

To calculate the entropy of thermal radiation we apply the first law of thermodynamics to radiation in an enclosure. Thus

$$T dS_{\text{rad}} = d\mathcal{E} + P dV = d(E^* V) + P^* dV \quad (69.3)$$

implies that

$$dS_{\text{rad}} = (4a_R T^2 V) dT + (\frac{4}{3} a_R T^3) dV = d(\frac{4}{3} a_R T^3 V). \quad (69.4)$$

Hence the entropy of equilibrium radiation is

$$S_{\text{rad}} = \frac{4}{3} a_R T^3 V. \quad (69.5)$$

If the volume of the cavity is changed adiabatically, $dS \equiv 0$, and from (69.4) we have

$$TV^{1/3} = \text{constant}, \quad (69.6)$$

which from (69.2) implies that

$$P^*V^{4/3} = \text{constant} \quad (69.7)$$

and hence

$$P^*T^{-4} = \text{constant}. \quad (69.8)$$

Comparison of these results with (4.13) to (4.15) shows that the polytropic laws for thermal radiation are identical to those for a perfect gas with $\gamma = \frac{4}{3}$, as mentioned in §43. Furthermore, from (14.19) to (14.21), with ρ replaced by V^{-1} , we find for thermal radiation that $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma = \frac{4}{3}$.

The heat capacity of radiation at constant volume is

$$C_v = (\partial \mathcal{E} / \partial T)_v = \partial(a_R T^4 V) / \partial T = 4a_R T^3 V. \quad (69.9)$$

However the heat capacity at constant pressure is *not* $C_p = \gamma C_v$, and in this sense the analogy between thermal radiation and a perfect gas with $\gamma = \frac{4}{3}$ fails. In fact, C_p is infinite. To understand this result physically, consider introducing heat into the enclosure while holding P^* constant. From (69.2), T remains constant, while V increases to accommodate the increase in energy of the system. Thus in (2.4), $dQ > 0$ while $dT \equiv 0$, hence $C_p = \infty$.

Finally, from (48.32) with $\Gamma_1 = \frac{4}{3}$ and $[p/(\hat{e} + p)]_{\text{rad}} = P^*/(E^* + P^*) = \frac{1}{4}$, we see that the speed of an ‘‘acoustic’’ disturbance in a gas of pure thermal radiation is $c/\sqrt{3}$.

70. Thermodynamics of Equilibrium Radiation Plus a Perfect Gas

Now consider a two-component gas comprising thermal radiation and a perfect gas of particles with mass $m = \mu_0 m_H$. The gas occupies a volume V and contains \mathcal{N} particles. To simplify the notation in this section and in §71 we write

$$p_g = p_{\text{gas}} = \mathcal{N}kT/V, \quad (70.1)$$

$$p = p_{\text{total}} = p_g + P^* = (\mathcal{N}kT/V) + \frac{1}{3}a_R T^4, \quad (70.2)$$

and define

$$\alpha \equiv P^*/p_g. \quad (70.3)$$

The total internal energy in the volume is

$$\mathcal{E} = \frac{3}{2}\mathcal{N}kT + a_R T^4 V, \quad (70.4)$$

hence the specific internal energy per unit mass is

$$e = \frac{3}{2}RT + (a_R T^4 / \rho). \quad (70.5)$$

To calculate the entropy of the system, we use the first law of thermodynamics

$$T dS = d\mathcal{E} + p dV. \quad (70.6)$$

Substituting (70.2) and (70.4) we find

$$dS = \mathcal{N}k d[\ln(T^{3/2}V)] + d\left(\frac{4}{3}a_R T^3 V\right), \quad (70.7)$$

which implies that

$$S = \mathcal{N}k \ln(T^{3/2}V) + \frac{4}{3}a_R T^3 V + \text{Constant}. \quad (70.8)$$

Equation (70.8) states that the total entropy of the composite gas equals the sum of the entropies of the radiation field and of the translational motion of the particles.

The heat capacity at constant volume is

$$\begin{aligned} C_v &= (\partial \mathcal{E} / \partial T)_v = \frac{3}{2}\mathcal{N}k + 4a_R T^3 V \\ &= \frac{3}{2}\mathcal{N}k(1 + 8\alpha) = 4a_R T^3 V[(1/8\alpha) + 1], \end{aligned} \quad (70.9)$$

which clearly yields the correct limits as $\alpha \rightarrow 0$ and $a \rightarrow \infty$. The specific heat at constant volume, per unit mass, is

$$c_v = \left(\frac{3}{2}k/m\right)(1 + 8\alpha). \quad (70.10)$$

The heat capacity at constant pressure follows from

$$C_p = (\partial \mathcal{H} / \partial T)_p, \quad (70.11)$$

where the total enthalpy is

$$\mathcal{H} = \mathcal{E} + pV = \frac{5}{2}\mathcal{N}kT + \frac{4}{3}a_R T^4 V. \quad (70.12)$$

Thus

$$C_p = \frac{5}{2}\mathcal{N}k + \frac{16}{3}a_R T^3 V + \frac{4}{3}a_R T^4 (\partial V / \partial T)_p. \quad (70.13)$$

From (70.3) one easily finds

$$(\partial V / \partial T)_p = (V/T)(1 + 4\alpha), \quad (70.14)$$

hence

$$C_p = \mathcal{N}k\left(\frac{5}{2} + 20\alpha + 16\alpha^2\right) \quad (70.15)$$

or

$$c_p = \frac{5}{2}(k/m)(1 + 8\alpha + \frac{32}{5}\alpha^2). \quad (70.16)$$

Equations (70.15) and (70.16) go to the correct limit as $\alpha \rightarrow 0$, and diverge as $\alpha \rightarrow \infty$, as expected from §69.

For an adiabatic change $dS = 0$, and (70.7) implies that

$$\left(\frac{3}{2}\frac{\mathcal{N}kT}{V} + 4a_R T^4\right)\left(\frac{V}{T}\right) dT_s + \left(\frac{\mathcal{N}kT}{V} + \frac{4}{3}a_R T^4\right) dV_s = 0, \quad (70.17)$$

whence

$$\begin{aligned} \Gamma_3 - 1 &\equiv -(\partial \ln T / \partial \ln V)_s = (p_g + 4P^*) / \left(\frac{3}{2}p_g + 12P^*\right) \\ &= (1 + 4\alpha) / \left(\frac{3}{2} + 12\alpha\right). \end{aligned} \quad (70.18)$$

From (70.3) we have

$$dp = (p_g + 4P^*)(dT/T) - p_g(dV/V), \quad (70.19)$$

hence

$$\begin{aligned}\Gamma_2/(\Gamma_2-1) &\equiv (\partial \ln p/\partial \ln T)_s \\ &= [1+4\alpha - (\partial \ln V/\partial \ln T)_s]/(1+\alpha).\end{aligned}\quad (70.20)$$

Using (70.18) in (70.20) we have

$$(\Gamma_2-1)/\Gamma_2 = (1+5\alpha+4\alpha^2)/(\frac{5}{2}+20\alpha+16\alpha^2). \quad (70.21)$$

In the limit as $\alpha \rightarrow 0$, (70.21) yields the same result as (4.15) for $\gamma = \frac{5}{3}$; as $\alpha \rightarrow \infty$ we recover (69.8) for pure radiation. To calculate Γ_1 we use (14.22), (70.18), and (70.21), obtaining

$$\Gamma_1 = (\frac{5}{2}+20\alpha+16\alpha^2)/[(\frac{5}{2}+12\alpha)(1+\alpha)]. \quad (70.22)$$

A table of Γ_1 , Γ_2 , and Γ_3 for values of $\beta \equiv p_g/p$ ranging from 0 to 1 is given in (C5, 59).

The speed of sound in the composite gas of material and thermal radiation can be computed from (48.32), with Γ_1 given by (70.22), p given by (70.2), and the total energy density by

$$\hat{e} = \rho_0 c^2 + (\mathcal{E}/V) \quad (70.23)$$

where $\rho_0 = Nm$ and \mathcal{E} is given by (70.4). It is easy to show that as $\alpha \rightarrow 0$ the speed of sound reduces to the adiabatic sound speed of the material, and as $\alpha \rightarrow \infty$ it approaches $c/\sqrt{3}$.

Finally, as in (14.31), we can define a variable mean molecular weight μ such that the total pressure (including radiation pressure) is given by $p = \rho kT/\mu m_H$; we can then calculate Q as defined in (14.33). For constant p ,

$$(\partial \ln \mu/\partial \ln T)_p = 1 + (\partial \ln \rho/\partial \ln T)_p, \quad (70.24)$$

hence

$$Q = -(\partial \ln \rho/\partial \ln T)_p. \quad (70.25)$$

From

$$p = (\rho kT/\mu_0 m_H) + \frac{1}{3} a_R T^4, \quad (70.26)$$

one easily finds

$$Q = 1 + 4\alpha. \quad (70.27)$$

Clearly $Q \rightarrow 1$ for a perfect gas ($\alpha \rightarrow 0$), and diverges for pure radiation ($\alpha \rightarrow \infty$).

71. Thermodynamics of Equilibrium Radiation Plus an Ionizing Gas

Let us now consider an equilibrium gas composed of thermal radiation and ionizing hydrogen. Writing x for the ionization fraction, the total pressure is

$$p = p_g + P^* = (1+x)(\mathcal{N}kT/V) + \frac{1}{3} a_R T^4, \quad (71.1)$$

and the total internal energy is

$$\mathcal{E} = \frac{3}{2}(1+x)\mathcal{N}kT + \mathcal{N}x\varepsilon_H + a_R T^4 V. \quad (71.2)$$

As in §14, x is determined from Saha's equation

$$x^2/(1-x) = \text{Const. } VT^{3/2} \exp(-\varepsilon_H/kT). \quad (71.3)$$

From (71.3) one finds

$$T \left(\frac{\partial x}{\partial T} \right)_v = \frac{x(1-x)}{(2-x)} \left(\frac{3}{2} + \frac{\varepsilon_H}{kT} \right), \quad (71.4)$$

and from (71.1) and (71.3) one can show that

$$T(\partial x/\partial T)_p = \frac{1}{2}x(1-x^2) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right] \quad (71.5)$$

and

$$(\partial \ln V/\partial \ln T)_p = 1 + 4\alpha + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]. \quad (71.6)$$

The heat capacity at constant volume is

$$C_v = \frac{3}{2}\mathcal{N}k(1+x) + \mathcal{N}k \left[\frac{3}{2} + (\varepsilon_H/kT) \right] T(\partial x/\partial T)_v + 4a_R T^3 V. \quad (71.7)$$

Using (71.4) in (71.7), we find the specific heat per unit mass, $c_v \equiv C_v/(\mathcal{N}m_H)$, is

$$c_v = \left(\frac{k}{m_H} \right) \left[\left(\frac{3}{2} + 12\alpha \right) (1+x) + \frac{x(1-x)}{(2-x)} \left(\frac{3}{2} + \frac{\varepsilon_H}{kT} \right)^2 \right], \quad (71.8)$$

which reduces to (14.15) when $\alpha = 0$, and to (70.10) when $x = 0$. It is evident that both ionization effects and radiation pressure can make a large contribution to c_v .

The heat capacity at constant pressure is obtained from (70.11), with

$$\mathcal{H} = \frac{5}{2}(1+x)\mathcal{N}kT + \mathcal{N}x\varepsilon_H + \frac{4}{3}a_R T^4 V. \quad (71.9)$$

Then

$$C_p = \frac{5}{2}\mathcal{N}k(1+x) + \mathcal{N}k \left[\frac{5}{2} + (\varepsilon_H/kT) \right] T(\partial x/\partial T)_p + \frac{4}{3}a_R T^3 [4V + T(\partial V/\partial T)_p]. \quad (71.10)$$

Using (71.5) and (71.6) in $c_p \equiv C_p/(\mathcal{N}m_H)$ we find, after some algebra,

$$c_p = (k/m_H) \left\{ \left(\frac{5}{2} + 20\alpha + 16\alpha^2 \right) (1+x) + \frac{1}{2}x(1-x^2) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]^2 \right\}, \quad (71.11)$$

which reduces to (14.18) when $\alpha = 0$ and to (70.16) when $x = 0$.

To compute adiabatic exponents we again require that

$$T dS = d\mathcal{E} + p dV \equiv 0. \quad (71.12)$$

Then, calculating $d\mathcal{E}$ from (71.2), $p dV$ from (71.1), and eliminating dx via the logarithmic derivative of (71.3),

$$\frac{(2-x)}{(1-x)} \frac{dx}{x} = \left(\frac{3}{2} + \frac{\varepsilon_H}{kT} \right) \frac{dT}{T} + \frac{dV}{V}, \quad (71.13)$$

one readily finds

$$\Gamma_3 - 1 = - \left(\frac{\partial \ln T}{\partial \ln V} \right)_s = \frac{1 + 4\alpha + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]}{\frac{3}{2} + 12\alpha + \frac{1}{2}x(1-x) \left\{ \left[\frac{3}{2} + (\varepsilon_H/kT) \right]^2 + \frac{3}{2} + 12\alpha \right\}} \quad (71.14)$$

which reduces to (70.18) when $x = 0$ and to (14.30) when $\alpha = 0$.

Using the equation of state along with (71.12) and (71.13) one can eliminate both dx and dV in favor of dp and dT . After considerable algebra one obtains

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \frac{1 + 4\alpha + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]}{\beta \left\{ \frac{5}{2} + 20\alpha + 16\alpha^2 + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]^2 \right\}}, \quad (71.15)$$

which reduces to (70.21) when $x = 0$ and to (14.28) when $\alpha = 0$. Next, from (14.22), (71.14), and (71.15) we find

$$\Gamma_1 = \frac{\beta \left\{ \frac{5}{2} + 20\alpha + 16\alpha^2 + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right]^2 \right\}}{\frac{3}{2} + 12\alpha + \frac{1}{2}x(1-x) \left\{ \left[\frac{3}{2} + (\varepsilon_H/kT) \right]^2 + \frac{3}{2} + 12\alpha \right\}}, \quad (71.16)$$

which reduces to (14.29) when $\alpha = 0$ and to (70.22) when $x = 0$.

Again, the speed of sound in the composite material-radiation gas can be computed from (48.32) with Γ_1 given by (71.16), p given by (71.1), and the total energy density \hat{e} by (70.27), with \mathcal{E} obtained from (71.2).

Finally, rewriting (70.24) as $Q = (\partial \ln V / \partial \ln T)_p$, we see from (71.6) that

$$Q = 1 + 4\alpha + \frac{1}{2}x(1-x) \left[\frac{5}{2} + (\varepsilon_H/kT) + 4\alpha \right], \quad (71.17)$$

which reduces to (14.34) and (70.26) in the appropriate limits.

More general formulae for the thermodynamic properties of a gas composed of thermal radiation and several ionizing species can be found in (C7, §9.18), (K2), and (M1).

The formulae derived in this section and in §70 give an accurate description of the thermodynamic properties of a radiating fluid when the radiation field is thermalized to its equilibrium distribution function and the material is in equilibrium at the same absolute temperature as the radiation. These formulae apply, for example, from the deeper layers of a stellar atmosphere down into the stellar interior. They sometimes can give useful first estimates even for a nonequilibrium radiation field, but in such cases they should be used with caution because not only may they be inaccurate numerically, but the whole conceptual framework of equilibrium thermodynamics on which they are based becomes problematical, or even invalid (§86).

6.3 The Interaction of Radiation and Matter

We now consider how radiation interacts with material. We first set forth formulae for computing rates of absorption, emission, and scattering of radiation in terms of atomic cross sections and level populations. As we

will see in §6.4 and §6.5, these quantities, if given, suffice to determine the radiation field via the equation of transfer.

72. Absorption, Emission, and Scattering

THE EXTINCTION COEFFICIENT

When radiation passes through material, energy is generally removed from the beam. We describe this loss in terms of an *opacity* or *extinction coefficient* (sometimes loosely called the *total absorption coefficient*) $\chi(\mathbf{x}, t; \mathbf{n}, \nu)$, defined such that an element of material of length dl and cross section dS , oriented normal to a beam of radiation having specific intensity $I(\mathbf{x}, t; \mathbf{n}, \nu)$ propagating along \mathbf{n} into solid angle $d\omega$ in frequency band $d\nu$, removes an amount of energy

$$\delta\mathcal{E} = \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu) dl dS d\omega d\nu dt \quad (72.1)$$

from the beam in a time interval dt . Opacity is the sum, over all states that can absorb at frequency ν , of the product of the occupation numbers of those states (cm^{-3}) times their atomic cross sections (cm^2) at that frequency. The dimensions of χ_ν are cm^{-1} ; the quantity $\lambda_\nu \equiv (1/\chi_\nu)$ cm is the *mean free path* of photons of frequency ν in the material.

In the fluid rest frame, the opacity is isotropic, but its frequency spectrum can be complicated, consisting of many overlapping continuum absorption edges, overlaid by thousands to millions of lines, each with a characteristic profile. In the laboratory frame, where the fluid is generally moving, the situation is much more complex. As a result of Doppler shift, a photon moving in direction \mathbf{n} with frequency ν in the lab frame has a frequency

$$\nu_0 = \nu(1 - \mathbf{n} \cdot \mathbf{v}/c) \quad (72.2)$$

in the fluid frame of material moving with velocity \mathbf{v} . Hence radiation moving in, say, the direction of the fluid flow interacts with the material at a different fluid-frame frequency than does radiation of the same lab-frame frequency moving in, say, the opposite direction. It is thus absorbed at a different rate because atomic cross sections vary with frequency; the lab-frame opacity therefore becomes anisotropic. (Strictly speaking we should also allow for the effects of *aberration* between the two frames; these can be ignored for our present purposes, but will be accounted for in Chapter 7.)

THE EMISSION COEFFICIENT

The *emission coefficient* (or *emissivity*) $\eta(\mathbf{x}, t; \mathbf{n}, \nu)$ of the material is defined such that the amount of radiant energy released by a material element of length dl and cross section dS , into a solid angle $d\omega$ around a direction \mathbf{n} , in frequency interval $d\nu$ in a time dt is

$$\delta\mathcal{E} = \eta(\mathbf{x}, t; \mathbf{n}, \nu) dl dS d\omega d\nu dt. \quad (72.3)$$

The dimensions of η are $\text{ergs cm}^{-3} \text{s}^{-1} \text{Hz}^{-1} \text{sr}^{-1}$. The emissivity may be isotropic in the rest frame of the material, but is anisotropic in the lab frame when the material moves, for the same reasons χ is.

We will sometimes add a subscript c for “continuum” and l for “line” to both χ and η .

SCATTERING

It is important to distinguish between “true” or “thermal” absorption-emission processes, and the process of *scattering*. In the former case, energy removed from the beam is converted into material thermal energy, and energy is emitted into the beam at the expense of material energy. Examples of “true absorption” processes are these: (1) A photon ionizes an atom; its energy goes into the ionization energy of the atom plus the kinetic energy of the free electron. (2) A photon excites an atom, which is subsequently de-excited by a collision with another particle; the photon’s energy goes into the kinetic energy of the collision partners. The inverses of these processes produce “thermal emission” in which energy is extracted from the thermal energy of hot material and converted into radiation. Other examples are given in (M2, §2.1).

In contrast, in a scattering process a photon interacts with a scattering center and emerges from the event moving in a different direction, generally with a slightly different frequency. Little or none of the photon’s energy goes into (or comes from) the thermal energy of the gas. Examples are as follows: (1) A photon excites an atom from state a to state b ; the atom decays radiatively back to state a . (2) A photon collides with a free electron (Thomson or Compton scattering) or with an atom or molecule in which it excites a resonance (Rayleigh or Raman scattering).

It is thus convenient to define a *true absorption coefficient* $\kappa(\mathbf{x}, t; \mathbf{n}, \nu)$ and a *scattering coefficient* $\sigma(\mathbf{x}, t; \mathbf{n}, \nu)$. The extinction coefficient is then

$$\chi(\mathbf{x}, t; \mathbf{n}, \nu) = \kappa(\mathbf{x}, t; \mathbf{n}, \nu) + \sigma(\mathbf{x}, t; \mathbf{n}, \nu). \quad (72.4)$$

Similarly we break the total emissivity into a thermal part η^t and a scattering part η^s :

$$\eta(\mathbf{x}, t; \mathbf{n}, \nu) = \eta^t(\mathbf{x}, t; \mathbf{n}, \nu) + \eta^s(\mathbf{x}, t; \mathbf{n}, \nu). \quad (72.5)$$

In certain simple situations we can write explicit expressions for η^s , which provide useful archetypes for later discussion. We will assume that the scattering is *conservative* so that all of the energy removed from the beam by the process is immediately re-emitted. For example, consider a spectrum line with total scattering cross section σ_s , and profile $\phi(\nu)$ normalized such that in the fluid frame

$$\int_0^\infty \phi(\nu_0) d\nu_0 = 1. \quad (72.6)$$

The suffix “0” on any quantity implies that it is measured in the comoving

frame. If σ_1 is isotropic, the total energy removed from the beam is

$$\begin{aligned} \sigma_1(\mathbf{x}, t) \int_0^\infty d\nu'_0 \phi(\mathbf{x}, t; \nu'_0) \oint d\omega'_0 I_0(\mathbf{x}, t; \mathbf{n}'_0, \nu'_0) \\ = 4\pi\sigma_1(\mathbf{x}, t) \int_0^\infty \phi(\mathbf{x}, t; \nu'_0) J_0(\mathbf{x}, t; \nu'_0) d\nu'_0. \end{aligned} \quad (72.7)$$

In general the re-emission of this energy is described by a *redistribution function* $R(\mathbf{n}', \nu'; \mathbf{n}, \nu)$ giving the joint probability that a photon (\mathbf{n}', ν') is absorbed and a photon (\mathbf{n}, ν) is emitted. We will not discuss the complication of *partial redistribution* [cf. (M2, Chaps. 2 and 13)], but will assume for simplicity that the photons are emitted *isotropically* in angle and are *randomly redistributed* (also called *complete redistribution*) over the line profile, in which case the fluid-frame emission by scattering is

$$\eta_0^s(\mathbf{x}, t; \nu_0) = \sigma_1(\mathbf{x}, t) \phi(\mathbf{x}, t; \nu_0) \int_0^\infty \phi(\mathbf{x}, t; \nu'_0) J_0(\mathbf{x}, t; \nu'_0) d\nu'_0. \quad (72.8)$$

In view of (72.2), the lab-frame emissivity is then

$$\eta^s(\mathbf{x}, t; \mathbf{n}, \nu) = \sigma_1(\mathbf{x}, t) \phi(\mathbf{x}, t; \mathbf{n}, \nu_0) \int_0^\infty d\nu' \oint d\omega' \phi(\mathbf{x}, t; \mathbf{n}', \nu'_0) I(\mathbf{x}, t; \mathbf{n}', \nu'), \quad (72.9)$$

where (again ignoring aberration)

$$\phi(\mathbf{x}, t; \mathbf{n}_0, \nu_0) \equiv \phi[\mathbf{x}, t; \mathbf{n}, \nu(1 - \mathbf{n} \cdot \mathbf{v}/c)]. \quad (72.10)$$

The assumption of complete redistribution is a good approximation in many cases of interest, for example, within the Doppler core of a line (where Doppler shifts efficiently scramble the frequencies of absorbed and emitted photons), or when excited atoms suffer many elastic collisions before a photon is re-emitted (the excited electrons are randomly redistributed over the substates of the upper level, destroying any correlation between absorption and emission frequencies in the line profile). The extreme opposite case occurs when the scattering is isotropic and *coherent*; then the emissivity is

$$\eta_0^s(\mathbf{x}, t; \nu_0) = \sigma_0(\mathbf{x}, t) J_0(\mathbf{x}, t; \nu_0). \quad (72.11)$$

This expression is often used to describe Thomson scattering of continuum photons by free electrons. One can assume coherence because the Thomson cross section σ_e is frequency independent, and the frequency variation of continuum radiation is slow enough that Doppler shifts produced by typical fluid velocities can be ignored. Similarly, isotropy is a good approximation because the angular variation of the (dipole) phase function is

weak. With these approximations (72.11) can be used in either the fluid or the lab frame.

We emphasize that the essential characteristic of scattering is that the rate of emission depends mainly on the radiation intensity at (\mathbf{x}, t) , and but little (if at all) on the amount of thermal energy there. Because the radiation field may originate mainly from *other* points in the medium, scattering processes are fundamentally *nonlocal*, and decouple the local emission rate from the local thermal pool. We also emphasize that (72.8), (72.9), and (72.11) are meant only to provide archetypes. In general, it is difficult to decide to what extent any particular process (e.g., absorption and emission of photons in a line) is a “true absorption-thermal emission” process or a “scattering” process. Most are a mixture because radiative and collisional processes operate simultaneously; this is true in both lines and continua. The true physics of the situation emerges only when the transfer equation is coupled directly to the equations of statistical equilibrium, which describe explicitly how atomic levels are populated and depopulated (see §6.6).

THE KIRCHHOFF–PLANCK RELATION

An important relation between thermal emission and absorption coefficients exists in strict thermodynamic equilibrium (TE). In an adiabatic enclosure, material (at rest) and radiation equilibrate to a uniform temperature and an isotropic radiation field [cf. (C5, 199–206) and (M5, 93–96)]. Moreover, in order to achieve a steady state, the amount of energy absorbed by the material in each range $(d\nu, d\omega)$ must exactly equal the amount it emits in that range. Therefore in TE

$$(\eta'_\nu)^* \equiv (\kappa_\nu I_\nu)^*, \quad (72.12)$$

where asterisks denote equilibrium values. But $I_\nu^* \equiv B_\nu(T)$, hence we obtain the *Kirchhoff-Planck relation*

$$(\eta'_\nu)^* \equiv \kappa_\nu^* B_\nu(T). \quad (72.13)$$

Strictly speaking, (72.13) applies only in TE. But when gradients of physical properties over a photon destruction length are very small, (72.13) is valid to a high degree of approximation at *local* values of the thermodynamic state variables. Hence we often invoke the hypothesis of local thermodynamic equilibrium (LTE) to write (in the comoving frame)

$$\eta'_0(\mathbf{x}, t; \nu_0) = \kappa_0^*(\mathbf{x}, t; \nu_0) B[\nu_0, T(\mathbf{x}, t)]. \quad (72.14)$$

Although (72.14) is certainly satisfactory in the diffusion limit (see §80) where the assumptions stated above hold, it cannot be guaranteed true, and may lead to significant errors, when free transport of radiation occurs, because the radiation field then acquires a nonlocal and/or nonequilibrium character that tends to drive the state of the material away from LTE. We shall analyze the meaning of LTE further in §84; in the meantime we

regard it as a computational expedient that sometimes must be used, even when of doubtful validity, to render a problem tractable (e.g., in most radiation hydrodynamics applications).

73. The Einstein Relations

Consider now radiative transitions between two bound atomic states: a lower level i with statistical weight g_i , and an upper level j with statistical weight g_j , which are separated by an energy $h\nu_{ij} = \varepsilon_j - \varepsilon_i$, where ε_i and ε_j are measured relative to the atom's ground state. Throughout this section all quantities are evaluated in the fluid frame.

The radiative processes that connect i and j are described by three probability coefficients B_{ij} , B_{ji} , and A_{ji} introduced by Einstein. The *absorption probability* B_{ij} is defined such that the number of photons absorbed in the line per unit volume per unit time is

$$r_{ij} = n_i \phi_\nu B_{ij} I_\nu (d\omega/4\pi) d\nu, \quad (73.1)$$

where ϕ_ν is the line profile. The rate of energy absorption per unit volume is then

$$a_\nu I_\nu = (B_{ij} h\nu_{ij}/4\pi) n_i \phi_\nu I_\nu. \quad (73.2)$$

Here a_ν is the macroscopic *absorption coefficient*, uncorrected for stimulated emission (see below).

An atom in the upper state can either decay *spontaneously* to the lower state, or be *stimulated* to decay by radiation in the line. The *spontaneous emission probability* A_{ji} is defined such that the rate of energy emission per unit volume is

$$\dot{\mathcal{E}}_\nu (\text{spontaneous}) = (A_{ji} h\nu_{ij}/4\pi) n_j \phi_\nu. \quad (73.3)$$

Here we have tacitly assumed that the line emission profile is identical to the absorption profile (complete redistribution). The *stimulated (or induced) emission probability* B_{ji} is defined such that the rate of stimulated energy emission per unit volume is

$$\dot{\mathcal{E}}_\nu (\text{stimulated}) = (B_{ji} h\nu_{ij}/4\pi) n_j \phi_\nu I_\nu. \quad (73.4)$$

Notice that spontaneous emission is isotropic, whereas stimulated emission has the same angular distribution as I_ν . In an induced emission, the incident photon leads to the emission of an *identical* photon (i.e., two photons emerge from the event). In this sense, induced emission can be viewed as *negative absorption*, and we can subtract (73.4) from (73.2) to obtain a *net absorption coefficient*, corrected for stimulated emission. This procedure is not quite correct because in general the absorption and emission profiles differ; however for complete redistribution they are identical.

The coefficients B_{ij} , B_{ji} , and A_{ji} are related, as can be seen by demanding

detailed balancing in thermodynamic equilibrium, which, from (73.2) to (73.4) implies

$$(n_i/n_j)^* B_{ij} I_\nu^* = A_{ji} + B_{ji} I_\nu^*, \quad (73.5)$$

where asterisks denote TE values. But $I_\nu^* \equiv B_\nu$, and by Boltzmann's formula (12.38)

$$(n_j/n_i)^* = (g_j/g_i) \exp(-h\nu_{ij}/kT), \quad (73.6)$$

hence

$$B_\nu = A_{ji}/[(n_i/n_j)^* B_{ij} - B_{ji}] = (A_{ji}/B_{ji})/[(g_i B_{ij}/g_j B_{ji}) e^{h\nu_{ij}/kT} - 1]. \quad (73.7)$$

Comparing (73.7) with (67.11) we see that

$$g_i B_{ij} = g_j B_{ji} \quad (73.8)$$

and

$$A_{ji} = (2h\nu_{ij}^3/c^2) B_{ji}. \quad (73.9)$$

Although our argument, for simplicity, invokes thermodynamic equilibrium, both (73.8) and (73.9) hold in general because the Einstein coefficients depend on atomic properties only.

From (73.2), (73.4), and (73.8) we can write the *line absorption coefficient* (in the comoving frame), corrected for stimulated emission as

$$\chi_l(\nu) = n_i (B_{ij} h\nu_{ij}/4\pi) [1 - (g_i n_j/g_j n_i)] \phi_\nu; \quad (73.10)$$

in the lab frame we must account for Doppler shifts in ϕ , as in (72.10). In LTE we can use (73.6) in (73.10) to obtain

$$\chi_l^*(\nu) = n_i^* (B_{ij} h\nu_{ij}/4\pi) [1 - \exp(h\nu_{ij}/kT)] \phi_\nu, \quad (73.11)$$

where n_i^* is computed from (13.6) using actual values of n_e and n_{ion} . The factor in square brackets in (73.11) is often called "the" correction for stimulated emission; however, this identification is correct only in LTE.

The *line emission coefficient* in the comoving frame is

$$\eta_l(\nu) = n_j (A_{ji} h\nu_{ij}/4\pi) \phi_\nu, \quad (73.12)$$

and the LTE emissivity is obtained by replacing n_j with n_j^* . In writing transfer equations (cf. §77) it is often convenient to use the ratio of emissivity to opacity, which is called the *source function* S_ν . For a line, the source function is

$$S_l = n_j A_{ji} / (n_i B_{ij} - n_j B_{ji}) = (2h\nu_{ij}^3/c^2) / [(g_i n_j/g_j n_i) - 1]. \quad (73.13)$$

Because the frequency variation of the factor ν^3 is weak compared to the variation of ϕ_ν , (73.13) is often called the *frequency-independent* line source function; in contrast, the line source function can have a very strong frequency dependence if we account for the difference between the emission and absorption profiles (partial redistribution). In LTE, S_l reduces to B_ν , as expected from the Kirchhoff–Planck relation.

74. The Einstein–Milne Relations

The Einstein relations were generalized to continua by Milne (M4), whose treatment we sketch; as in §73 we work in the fluid frame. Suppose an atom is photoionized to produce an ion plus a free electron moving with speed v . Let n_0 be the number density of atoms, n_1 the density of ions, and $n_e(v) dv$ the density of electrons with speeds on the range $(v, v+dv)$, assumed Maxwellian. If p_ν is the *photoionization probability* of the atom by radiation in the frequency range $(\nu, \nu+dv)$, the *photoionization rate* is $n_0 p_\nu I_\nu dv$; the energy absorption coefficient is $\alpha_\nu = h\nu p_\nu$. Let $F(\nu)$ be the *spontaneous recombination probability* and $G(\nu)$ the *induced recombination probability* for electrons with speeds $(v, v+dv)$ to recombine with the ions. Then the *recombination rate* for electrons with speed v is $n_1 n_e(v) [F(\nu) + G(\nu) I_\nu] v dv$. The photon energy required to ionize the atom and produce an electron with speed v is

$$h\nu = \varepsilon_{\text{ion}} + \frac{1}{2}mv^2, \quad (74.1)$$

whence we have $h dv = mv dv$.

In thermodynamic equilibrium, the number of photoionizations equals the number of recombinations. Therefore

$$n_0^* p_\nu B_\nu = n_1^* n_e(v) [F(\nu) + G(\nu) B_\nu] (h/m), \quad (74.2)$$

which implies that

$$B_\nu = [F(\nu)/G(\nu)] / \{ [n_0^* p_\nu m / n_1^* n_e(v) h G(\nu)] - 1 \}. \quad (74.3)$$

Comparing (74.3) with (67.11) we see that

$$F(\nu) = (2h\nu^3/c^2) G(\nu) \quad (74.4)$$

and

$$p_\nu / G(\nu) = (h/m) [n_e(v) (n_1/n_0)^*] e^{h\nu/kT}. \quad (74.5)$$

But in TE, $n_e(v) dv$ is the Maxwellian distribution

$$n_e(v) dv = n_e (m/2\pi kT)^{3/2} \exp(-\frac{1}{2}mv^2/kT) 4\pi v^2 dv, \quad (74.6)$$

and the ratio $(n_1 n_e/n_0)^*$ is given by Saha's equation

$$(n_1/n_0)^* = n_e (g_0/2g_1) (h^2/2\pi m kT)^{3/2} \exp(\varepsilon_{\text{ion}}/kT) \equiv n_e \Phi_0(T). \quad (74.7)$$

Using (74.1), (74.6), and (74.7) in (74.5) we obtain

$$p_\nu = (8\pi m^2 v^2 g_1 / h^2 g_0) G(\nu) = (4\pi c^2 m^2 v^2 g_1 / h^3 g_0 \nu^3) F(\nu), \quad (74.8)$$

where the second equality follows from (74.4). Equations (74.4) and (74.8) are the continuum analogues of (73.8) and (73.9); they apply in general, not just in TE.

Using the above results we can write the continuum absorption coefficient, corrected for stimulated emission, as

$$\kappa_\nu = h\nu [n_0 p_\nu - (h/m) n_1 n_e(v) G(\nu)]. \quad (74.9)$$

Recalling that $\alpha_\nu = h\nu p_\nu$, and using (74.1) and (74.6) to (74.8) we find

$$\kappa_\nu = (n_0 - n_0^* e^{-h\nu/kT}) \alpha_\nu, \quad (74.10)$$

where n_0^* is the LTE value of n_0 computed from Saha's equation using actual values of n_e and n_1 , that is, $n_0^* = n_1 n_e \Phi_0(T)$. In LTE,

$$\kappa_\nu^* = n_0^* (1 - e^{-h\nu/kT}) \alpha_\nu. \quad (74.11)$$

As before the factor $(1 - e^{-h\nu/kT})$ is often called the correction factor for stimulated emission, but this is correct only in LTE. Notice that in the continuum the induced emission rate always has its LTE value, whereas for a spectral line this rate depends on the actual upper level population, and hence may depart from its LTE value. This is not surprising, because recombination, whether spontaneous or induced, results from collisions between ions and electrons; if these particles have an equilibrium (Maxwellian) velocity distribution, recombination must occur at the LTE rate.

The spontaneous continuum emission coefficient is

$$\eta'_\nu = h\nu n_1 n_e(v) F(v) (h/m) = [hn_1 n_e(v) F(v) / mp_\nu] \alpha_\nu. \quad (74.12)$$

Using (74.6) to (74.8) we find that (74.12) reduces to

$$\eta'_\nu = (2h\nu^3/c^2) n_0^* \alpha_\nu e^{-h\nu/kT} = n_0^* (1 - e^{-h\nu/kT}) \alpha_\nu B_\nu(T) = \kappa_\nu^* B_\nu(T). \quad (74.13)$$

Thus, provided that we define n_0^* in terms of the actual density of electrons and ions, continuum emission occurs at its LTE rate (as predicted by the Kirchoff-Planck relation) because it is a collisional process. Hence for continua, the general formula for the opacity differs from its LTE form, but that for the emissivity does not; for lines, the general opacity and emissivity both differ from their LTE forms.

75. Opacity and Emission Coefficients

In addition to the *bound-bound* (line) and *bound-free* (photoionization) processes described in §§73 and 74, radiation can be absorbed and emitted during collisions between two free particles in *free-free* transitions (e.g., bremsstrahlung). Because this process is collisional, it always occurs at the LTE rate (using actual electron and ion densities). The total opacity (emissivity) at any frequency ν is the sum of the opacities (emissivities) of all processes that occur at the frequency. If we write $\alpha_{ij}(\nu)$, $\alpha_{i\kappa}(\nu)$, and $\alpha_{\kappa\kappa}(\nu)$ for bound-bound, bound-free, and free-free cross sections respectively, then from (73.10) and (74.10) the *total opacity* is

$$\begin{aligned} \chi_\nu &= \sum_i \sum_{j>i} [n_i - (g_i/g_j) n_j] \alpha_{ij}(\nu) + \sum_i (n_i - n_i^* e^{-h\nu/kT}) \alpha_{i\kappa}(\nu) \\ &\quad + \sum_\kappa n_e n_\kappa \alpha_{\kappa\kappa}(\nu, T) (1 - e^{-h\nu/kT}) + n_e \sigma_e \\ &\equiv \kappa_\nu + \sigma_\nu, \end{aligned} \quad (75.1)$$

where the last term represents Thomson scattering by free electrons.

Similarly, from (73.12) and (74.13) the *total thermal emissivity* is

$$\eta'_\nu = (2h\nu^3/c^2) \left[\sum_i \sum_{j>i} n_j (g_i/g_j) \alpha_{ij}(\nu) + \sum_i n_i^* \alpha_{i\kappa}(\nu) e^{-h\nu/kT} + \sum_\kappa n_e n_\kappa \alpha_{\kappa\kappa}(\nu, T) e^{-h\nu/kT} \right] \quad (75.2)$$

Both (75.1) and (75.2) apply in the fluid frame. If the fluid moves, we must account for Doppler shifts as in (72.2) when calculating lab-frame opacities and emissivities, and in general we have $\chi = \chi(\mathbf{x}, t; \mathbf{n}, \nu)$ and $\eta = \eta(\mathbf{x}, t; \mathbf{n}, \nu)$ in the lab frame.

In the limit of LTE, (75.1) and (75.2) simplify to

$$\chi_\nu^* = \left\{ \sum_i n_i^* \left[\alpha_{i\kappa}(\nu) + \sum_{j>i} \alpha_{ij}(\nu) \right] + \sum_\kappa n_e n_\kappa \alpha_{\kappa\kappa}(\nu, T) \right\} (1 - e^{-h\nu/kT}) + n_e \sigma_e \quad (75.3)$$

$$\equiv \kappa_\nu^* + n_e \sigma_e,$$

and

$$(\eta'_\nu)^* = (2h\nu^3/c^2) e^{-h\nu/kT} \left\{ \sum_i n_i^* \left[\alpha_{i\kappa}(\nu) + \sum_{j>i} \alpha_{ij}(\nu) \right] + \sum_\kappa n_e n_\kappa \alpha_{\kappa\kappa}(\nu, T) \right\}. \quad (75.4)$$

Clearly $(\eta'_\nu)^* = \kappa_\nu^* B_\nu$, as expected from the Kirchhoff–Planck relation (72.11). Again (75.3) and (75.4) apply in the comoving frame; in the lab frame both χ^* and η^* depend on $(\mathbf{x}, t; \mathbf{n}, \nu)$ when the fluid moves.

6.4 The Equation of Transfer

76. Derivation of the Transfer Equation

Consider an element of material of length ds and cross section dS , fixed in the laboratory frame. We calculate the change, in a time dt , in the energy of the radiation field contained in a frequency interval $d\nu$, traveling into solid angle $d\omega$ along a direction \mathbf{n} normal to dS , as it passes through the material (see Figure 76.1). The difference between the amount of energy that emerges at position $\mathbf{x} + \Delta\mathbf{x}$ at time $t + \Delta t$ and the amount incident at position \mathbf{x} at time t must equal the difference between the amount of energy created by emission from the material and the amount absorbed. Thus in a Cartesian coordinate system

$$[I(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t; \mathbf{n}, \nu) - I(\mathbf{x}, t; \mathbf{n}, \nu)] dS d\omega d\nu dt = [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu)] ds dS d\omega d\nu dt. \quad (76.1)$$

If we let s be the path length along the ray, $\Delta t = ds/c$, and

$$I(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t; \mathbf{n}, \nu) = I(\mathbf{x}, t; \mathbf{n}, \nu) + [(1/c)(\partial I/\partial t) + (\partial I/\partial s)] ds. \quad (76.2)$$

Substituting (76.2) into (76.1) we obtain the *transfer equation*

$$[(1/c)(\partial/\partial t) + (\partial/\partial s)] I(\mathbf{x}, t; \mathbf{n}, \nu) = \eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu). \quad (76.3)$$

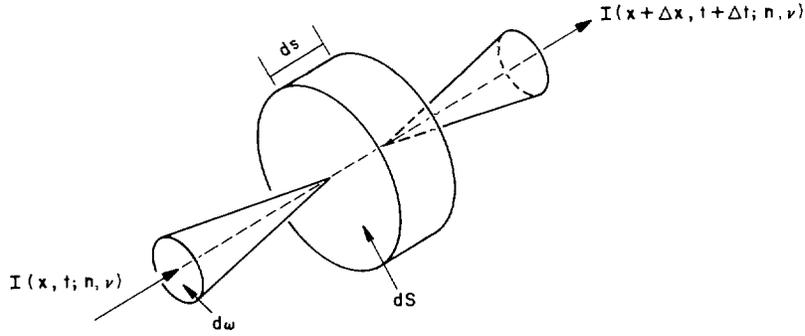


Fig. 76.1 Pencil of radiation passing through a material element.

Because s is a coordinate-independent pathlength, (76.3) applies in arbitrary coordinate systems, provided we use an appropriate expression to evaluate $(\partial/\partial s)$.

The derivation just given of the transfer equation is classical, macroscopic, and phenomenological in character. It omits reference to such important phenomena as polarization, dispersion, coherence, interference, and quantum effects, none of which are correctly described by (76.3). An excellent discussion of the approximations inherent in, and the validity of, the classical radiative transfer equation is given in (P3, 47–49). Good discussions of the transfer equation from the point of view of quantum field theory are given in (H1), (L1), (L2), (L3), (O1).

The mathematical expression for $(\partial/\partial s)$ depends on geometry. In Cartesian coordinates

$$\frac{\partial I}{\partial s} = \left(\frac{\partial x}{\partial s}\right) \frac{\partial I}{\partial x} + \left(\frac{\partial y}{\partial s}\right) \frac{\partial I}{\partial y} + \left(\frac{\partial z}{\partial s}\right) \frac{\partial I}{\partial z} = n_x \frac{\partial I}{\partial x} + n_y \frac{\partial I}{\partial y} + n_z \frac{\partial I}{\partial z}, \quad (76.4)$$

where (n_x, n_y, n_z) are components of the unit vector \mathbf{n} along the direction of propagation. The transfer equation is then

$$[(1/c)(\partial/\partial t) + (\mathbf{n} \cdot \nabla)]I(\mathbf{x}, t; \mathbf{n}, \nu) = \eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu)I(\mathbf{x}, t; \mathbf{n}, \nu). \quad (76.5)$$

For a one-dimensional planar atmosphere, (76.5) reduces to

$$[(1/c)(\partial/\partial t) + \mu(\partial/\partial z)]I(z, t; \mu, \nu) = \eta(z, t; \mu, \nu) - \chi(z, t; \mu, \nu)I(z, t; \mu, \nu), \quad (76.6)$$

and for static media or steady flows the time derivative can be dropped, yielding

$$\mu[\partial I(z; \mu, \nu)/\partial z] = \eta(z; \mu, \nu) - \chi(z; \mu, \nu)I(z; \mu, \nu). \quad (76.7)$$

If the opacity and emissivity are given, (76.7) is an ordinary differential equation, while (76.6) is a partial differential equation. If scattering terms

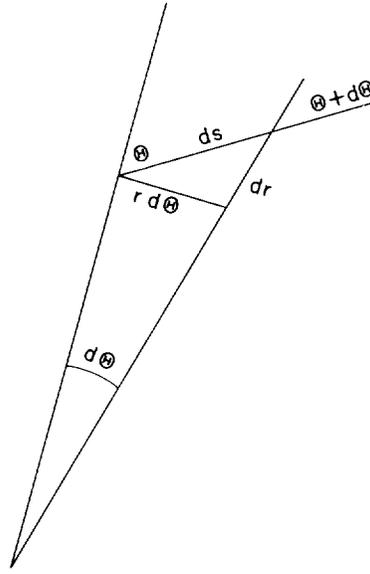


Fig. 76.2 Photon propagation angle in spherical symmetry.

are present, the mean intensity (an integral over angle) appears on the right-hand side, producing *integrodifferential equations*.

In curvilinear coordinates the coordinate basis vectors rotate with respect to the straight-line path determined by a fixed propagation vector \mathbf{n} . Therefore to allow for the changes in the components of \mathbf{n} measured along these basis vectors we evaluate $(\partial/\partial s)$ as $(\partial/\partial s) = \mathbf{n} \cdot \nabla + (d\mathbf{n}/ds) \cdot \nabla_{\mathbf{n}}$ where $\nabla_{\mathbf{n}}$ denotes differentiation with respect to the direction cosines of \mathbf{n} . For example, in general spherical geometry, the derivative $(\partial/\partial s)$ introduces terms in $\partial/\partial r$, $\partial/\partial \theta$, $\partial/\partial \phi$, $\partial/\partial \Theta$, and $\partial/\partial \Phi$. But for spherical symmetry the terms $\partial/\partial \theta$, $\partial/\partial \phi$, and $\partial/\partial \Phi$ all vanish identically. From Figure 76.2 we see that $dr = \cos \Theta ds = \mu ds$ and $r d\Theta = -\sin \Theta ds = -(1 - \mu^2)^{1/2} ds$, hence

$$\frac{\partial}{\partial s} = \left(\frac{\partial r}{\partial s}\right) \frac{\partial}{\partial r} + \left(\frac{\partial \Theta}{\partial s}\right) \frac{\partial}{\partial \Theta} = \cos \Theta \frac{\partial}{\partial r} - \frac{\sin \Theta}{r} \frac{\partial}{\partial \Theta} = \mu \frac{\partial}{\partial r} + \frac{(1 - \mu^2)^{1/2}}{r} \frac{\partial}{\partial \mu}. \quad (76.8)$$

Therefore the transfer equation for a spherically symmetric medium is

$$\left[\frac{1}{c} \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{(1 - \mu^2)^{1/2}}{r} \frac{\partial}{\partial \mu} \right] I(r, t; \mu, \nu) = \eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu). \quad (76.9)$$

Notice that even when the $\partial/\partial t$ term is dropped, (76.9) is a partial differential equation, or a partial integrodifferential equation.

77. Optical Depth and Source Function

The concept of *optical depth* is central to discussions of transfer. If \mathbf{x} and \mathbf{x}' are two points in the medium separated by $l = |\mathbf{x}' - \mathbf{x}|$, the optical depth between them is

$$\tau_\nu(\mathbf{x}, \mathbf{x}') = \int_0^l \chi(\mathbf{x} + \mathbf{n}s; \mathbf{n}, \nu) ds \quad (77.1)$$

where ds is a path-length increment, and \mathbf{n} is a unit vector along the straight line $(\mathbf{x}, \mathbf{x}')$. Here we have allowed for the possibility that the material may be moving, in which case τ can depend both on the separation between the two points, and on the direction in which the integration is performed. Recalling that χ_ν^{-1} is the mean free path of a photon of frequency ν , we see that $\tau_\nu(\mathbf{x}, \mathbf{x}')$ is equal to the number of photon mean free paths between \mathbf{x} and \mathbf{x}' .

For a static planar medium, optical depth is customarily measured vertically downward from the upper boundary at $z = z_{\max}$, and usually provides a more convenient depth variable for transfer calculations than does the geometrical depth z . Remembering that z increases upward we have

$$d\tau_\nu = -\chi_\nu dz \quad (77.2)$$

and

$$\tau_\nu(z) = \int_z^{z_{\max}} \chi_\nu(z') dz'. \quad (77.3)$$

For a slant ray emerging from the medium with angle-cosine μ relative to the vertical, $ds = dz/\mu$, hence the slant optical depth along the ray down to geometric depth z is $\tau_\nu(z)/\mu$. For static spherical media a similar definition can be written for the *radial optical depth*, that is, the optical depth measured inward along a radius vector.

The transfer equation is often written in terms of the source function

$$S(\mathbf{x}, t; \mathbf{n}, \nu) \equiv \eta(\mathbf{x}, t; \mathbf{n}, \nu) / \chi(\mathbf{x}, t; \mathbf{n}, \nu). \quad (77.4)$$

For example, in a static planar medium the transfer equation assumes its "standard form"

$$\mu(\partial I_\nu / \partial \tau_\nu) = I_\nu - S_\nu; \quad (77.5)$$

we study this equation extensively in §6.5.

The discussion in §6.3 suggests several archetype expressions for S_ν ; unless specified otherwise these apply only in the comoving frame of the fluid, or in static media. In LTE, (72.13) implies

$$S_\nu = B_\nu(T). \quad (77.6)$$

If we have a contribution from coherent isotropic scattering then

$$\chi_\nu = \kappa_\nu + \sigma_\nu \quad (77.7a)$$

and
$$\eta_\nu = \kappa_\nu B_\nu + \sigma_\nu J_\nu, \quad (77.7b)$$

hence
$$S_\nu = (\kappa_\nu B_\nu + \sigma_\nu J_\nu) / (\kappa_\nu + \sigma_\nu). \quad (77.8)$$

For a spectrum line with an overlapping LTE continuum

$$\chi_\nu = \kappa_c + \chi_l \phi_\nu. \quad (77.9)$$

If a fraction ε of the line emission is thermal and the remainder is isotropic scattering with complete redistribution, then

$$\eta_\nu = \kappa_c B_\nu + \chi_l \phi_\nu \left[(1 - \varepsilon) \int \phi_\nu J_\nu d\nu + \varepsilon B_\nu \right], \quad (77.10)$$

hence

$$S_\nu = \left(\frac{r + \varepsilon \phi_\nu}{r + \phi_\nu} \right) B_\nu + \left[\frac{(1 - \varepsilon) \phi_\nu}{r + \phi_\nu} \right] \int \phi_\nu J_\nu d\nu \equiv \xi_\nu B_\nu + (1 - \xi_\nu) \bar{J}, \quad (77.11)$$

where $r \equiv \kappa_c / \chi_l$. We give a physical justification for (77.10) and (77.11) in §87. In the laboratory frame, ϕ_ν becomes $\phi(\mathbf{x}, t; \mathbf{n}, \nu)$ as in (72.10), and \bar{J} becomes a double integral of the specific intensity over both angle and frequency, as in (72.9).

We emphasize that the source functions (77.6), (77.8), and (77.11) are meant only to be illustrative; a more complete discussion is given in §6.7.

78. Moments of the Transfer Equation

Angular moments of the transfer equation are both physically important and mathematically useful. To obtain the *zero-order moment equation* we multiply the time-dependent transfer equation in Cartesian coordinates,

$$[(1/c)(\partial/\partial t) + n^i(\partial/\partial x^i)]I(\mathbf{x}, t; \mathbf{n}, \nu) = \eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu)I(\mathbf{x}, t; \mathbf{n}, \nu), \quad (78.1)$$

by $(d\omega/4\pi)$ and integrate over all solid angles. Using (64.1), (65.2), and (65.6) we find

$$(1/c)(\partial J_\nu/\partial t) + \nabla \cdot \mathbf{H}_\nu = (1/4\pi) \oint [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu)I(\mathbf{x}, t; \mathbf{n}, \nu)] d\omega; \quad (78.2)$$

or, in view of (64.4),

$$(\partial E_\nu/\partial t) + \nabla \cdot \mathbf{F}_\nu = \oint [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu)I(\mathbf{x}, t; \mathbf{n}, \nu)] d\omega \quad (78.3)$$

Integrating over all frequencies we have

$$(\partial E/\partial t) + \nabla \cdot \mathbf{F} = \int_0^\infty d\nu \oint d\omega [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu)I(\mathbf{x}, t; \mathbf{n}, \nu)]. \quad (78.4)$$

The reduction of these equations to one-dimensional planar geometry is trivial.

Equations (78.3) and (78.4) are *energy equations* for the radiation field. Integrating them over a fixed volume element and applying the divergence

theorem we see that the rate of change of the radiant energy in the volume equals (1) the total rate of energy emission from the material, minus (2) the total rate of energy absorption by the material, minus (3) the net flow of radiant energy through the volume element's boundary surface.

For a spherically symmetric medium, (78.3) and (78.4) become

$$\begin{aligned} (\partial E_\nu / \partial t) + r^{-2} [\partial(r^2 F_\nu) / \partial r] \\ = 2\pi \int_{-1}^1 [\eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu)] d\mu, \end{aligned} \quad (78.5)$$

and

$$\begin{aligned} (\partial E / \partial t) + r^{-2} [\partial(r^2 F) / \partial r] \\ = 2\pi \int_0^\infty d\nu \int_{-1}^1 d\mu [\eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu)], \end{aligned} \quad (78.6)$$

which can also be obtained by direct integration of (76.9) over $d\omega$ and $d\nu$. The total *luminosity* passing through a spherical shell of radius r is

$$L(r, t) = 4\pi r^2 F(r, t), \quad (78.7)$$

so (78.6) can be rewritten as

$$\begin{aligned} (\partial E / \partial t) + (4\pi r^2)^{-1} (\partial L / \partial r) \\ = 2\pi \int_0^\infty d\nu \int_{-1}^1 d\mu [\eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu)]. \end{aligned} \quad (78.8)$$

In a *static* medium (i.e., no time-dependence or hydrodynamic motions) we must have $(\partial E / \partial t) \equiv 0$. Furthermore, for the material to be in a steady state it must be in *radiative equilibrium* (i.e., it must emit exactly as much energy as it absorbs). Under these conditions the right-hand sides of (78.4) and (78.8) vanish identically, hence $\nabla \cdot \mathbf{F} \equiv 0$. That is, in radiative equilibrium the flux is constant with depth in planar geometry, and the luminosity is constant with radius in spherical geometry. We emphasize that radiative equilibrium occurs only in an absolutely static medium, and represents a limiting form of the radiation energy equation. We will discuss the general radiation energy equation and its coupling to energy equations for radiating fluids in motion in greater detail in Chapter 7.

The *first-order moment equation* for the radiation field is obtained by multiplying (78.1) by \mathbf{n} , and integrating against $(d\omega/4\pi)$, which yields

$$\begin{aligned} c^{-1} (\partial H_\nu^i / \partial t) + (\partial K_\nu^{ij} / \partial x^j) = (1/4\pi) \oint [\eta(\mathbf{x}, t; \mathbf{n}, \nu) \\ - \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu)] n^i d\omega. \end{aligned} \quad (78.9)$$

Here $K_\nu^{ij} \equiv (c/4\pi) P_\nu^{ij}$, as defined by (66.2). Multiplying (78.9) by $(4\pi/c)$ we obtain, in tensor notation

$$c^{-2} (\partial \mathbf{F}_\nu / \partial t) + \nabla \cdot \mathbf{P}_\nu = c^{-1} \oint [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu)] \mathbf{n} d\omega, \quad (78.10)$$

which, integrated over all frequencies, yields

$$c^{-2}(\partial \mathbf{F}/\partial t) + \nabla \cdot \mathbf{P} = c^{-1} \int_0^\infty d\nu \oint d\omega [\eta(\mathbf{x}, t; \mathbf{n}, \nu) - \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu)] \mathbf{n}. \quad (78.11)$$

Equations (78.10) and (78.11) are *momentum equations* for the radiation field. To verify this interpretation, recall from (65.7) and (65.8) that the radiative momentum density is c^{-2} times the flux, and from §66 that \mathbf{P} is the radiation-momentum flux-density tensor. Furthermore, the momentum of a photon with energy $h\nu$ moving in direction \mathbf{n} is $(h\nu/c)\mathbf{n}$. Thus integrating (78.11) over a fixed volume, and applying the divergence theorem, we find that the rate of change of the radiant momentum in the volume equals (1) the net rate of momentum input into the radiation field by emission from the material, minus (2) the net rate of absorption of radiation momentum by the material, minus (3) the rate of transport of radiative momentum across the boundary surface of the volume. As a by-product we see that the integral

$$\mathbf{f}_R = c^{-1} \int_0^\infty d\nu \oint d\omega \chi(\mathbf{x}, t; \mathbf{n}, \nu) I(\mathbf{x}, t; \mathbf{n}, \nu) \mathbf{n} \quad (78.12)$$

is the *radiation force*, per unit volume, on the material.

In a spherically symmetric medium, (78.10) and (78.11) become

$$\begin{aligned} c^{-2}(\partial F_\nu/\partial t) + (\partial P_\nu/\partial r) + (3P_\nu - E_\nu)/r \\ = (2\pi/c) \int_{-1}^1 [\eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu)] \mu d\mu, \end{aligned} \quad (78.13)$$

and

$$\begin{aligned} c^{-2}(\partial F/\partial t) + (\partial P/\partial r) + (3P - E)/r \\ = (2\pi/c) \int_0^\infty d\nu \int_{-1}^1 d\mu [\eta(r, t; \mu, \nu) - \chi(r, t; \mu, \nu) I(r, t; \mu, \nu)] \mu. \end{aligned} \quad (78.14)$$

These results can also be obtained by direct integration (76.9) over $\mu d\omega$ and $d\nu$.

Thus far we have allowed for material motions, assuming that χ and η depend on angle and frequency. Considerable simplification is obtained for a static medium ($\mathbf{v} \equiv 0$) when, in addition, the radiation field is time independent. These assumptions provide a good framework for the development of basic methods for solving transfer equations (cf. §6.5). Because both χ and η are isotropic in a static medium, (78.2) in planar geometry reduces to

$$(\partial H_\nu/\partial z) = \eta_\nu - \chi_\nu J_\nu, \quad (78.15a)$$

or

$$(\partial F_\nu/\partial z) = 4\pi\eta_\nu - c\chi_\nu E_\nu. \quad (78.15b)$$

Similarly (78.10) reduces to

$$(\partial K_\nu/\partial z) = [\partial(f_\nu J_\nu)/\partial z] = -\chi_\nu H_\nu \quad (78.16a)$$

or

$$(\partial P_\nu/\partial z) = [\partial(f_\nu E_\nu)/\partial z] = -(\chi_\nu/c)F_\nu. \quad (78.16b)$$

The radiation momentum equation now reads

$$(\partial P/\partial z) = -c^{-1} \int_0^\infty \chi_\nu F_\nu d\nu; \quad (78.17)$$

the integral over η_ν vanishes because the net momentum loss by the material through isotropic emission is identically zero.

Similarly, in a static spherical medium, (78.5) reduces to

$$r^{-2}[\partial(r^2 F_\nu)/\partial r] = 4\pi\eta_\nu - c\chi_\nu E_\nu, \quad (78.18)$$

and (78.13) becomes

$$(\partial P_\nu/\partial r) + (3P_\nu - E_\nu)/r = -(\chi_\nu/c)F_\nu \quad (78.19a)$$

or

$$\partial(f_\nu E_\nu)/\partial r + (3f_\nu - 1)E_\nu/r = -(\chi_\nu/c)F_\nu. \quad (78.19b)$$

In addition to their physical significance, the moment equations provide powerful tools for solving transfer problems because they eliminate angle variables from the problem and thereby reduce its dimensionality. On the other hand, from (78.15) and (78.16), or (78.18) and (78.19), we see an essential difficulty: the first n moment equations always contain all moments through order $n+1$; thus we have one more unknown to determine than there are equations. This difficulty is known as the *closure problem*. It is very instructive to compare the closure problem for the radiation equations with the corresponding problem for the equations of gas dynamics. Inasmuch as the specific intensity characterizes fully both the angular and energy distribution of the radiation field, our derivation of the radiation energy and momentum equations in terms of moments of the intensity is conceptually identical to the derivation of the fluid equations from kinetic theory as discussed in §30 (and in §43 for relativistic fluids). We saw there that we can write the energy density, heat flux, and stress (including viscous effects) in the fluid in terms of suitable averages over the distribution function. If, as in §31, we assume that the distribution function is isotropic, the system of fluid equations closes exactly, and both the heat flux and the viscous stresses vanish identically. The same is true for radiation; if we assume I_ν is perfectly isotropic, we know that the radiation stress tensor becomes diagonal and isotropic with $P_\nu = \frac{1}{3}E_\nu$, and that $F_\nu \equiv 0$, so no further closure is necessary. On the other hand, if we assume that the distribution function is not isotropic, but that $\lambda/l \ll 1$, where λ is a particle mean free path, we are again able to achieve closure by deriving explicit expressions for the fluid heat flux \mathbf{q} and the viscous stress tensor $\boldsymbol{\sigma}$. As is

shown in §§80 and 97, entirely analogous results are possible for radiation in the limit that $\lambda_p/l \ll 1$, where λ_p is the photon mean free path.

The real problem arises near boundary surfaces where a mean free path (photon or particle) may exceed any characteristic structural length in the flow. We must then find other methods for evaluating the averages that appear as the energy flux or as nonisotropic (perhaps even off-diagonal) contributions to the stress tensor, in the fluid and/or radiation energy and momentum equations. We have ignored this problem for ordinary fluids because it becomes important only in extremely rarefied flows [e.g., the interplanetary medium **(H3)**]. But it cannot be ignored for radiation because we always must deal with regions in which $\lambda/l \ll 1$ while $\lambda_p/l \gg 1$; indeed these are the very layers of a radiating flow that we can observe. Here we must face the closure problem squarely.

In one-dimensional problems we have two equations containing the three scalars E_ν , F_ν , and P_ν , and one approach is to close the system with variable Eddington factors f_ν , as in (78.16b) and (78.19b). When solving the moment equations we assume that f_ν is known. We subsequently determine f_ν from a separate angle-by-angle formal solution of the full transfer equation assuming that the radiation energy density (which appears in the source function) is known; we then iterate the two steps to convergence. As the value of f_ν converges, the closure becomes essentially exact. In radiation-hydrodynamics calculations where computational speed is paramount, a yet-simpler procedure is sometimes adopted: one uses approximate analytical formulae to determine f from the geometry of the problem and from that ratio (F/E) [see, e.g., **(F2)**, **(S4)**]. G. Minerbo **(M6)** developed an elegant formulation of this kind; Minerbo's formulation is useful also in multidimensional problems where the full Eddington tensor $f \equiv P/E$ must be specified.

Alternatively, we can rewrite the transfer equation in terms of angle-dependent mean-intensity-like and flux-like variables (see §83), and obtain exact closure of two coupled angle-dependent equations that strongly resemble the moment equations, and have many of their desirable properties. These equations can be discretized and solved directly by efficient numerical methods.

6.5 Solution of the Transfer Equation

We now address the problem of *solving* the transfer equation. To develop insight we first discuss the formal solution and special solutions for important special cases; we then discuss general numerical techniques. Inasmuch as we now focus mainly on mathematical rather than physical content of the equations, we will usually use the Eddington variables J_ν , H_ν , and K_ν in preference to the dynamical variables E_ν , F_ν , and P_ν .

We concentrate almost entirely on the solution of the *time-independent* transfer equation (the exception is an analytical expression for the time-dependent formal solution). The techniques developed here provide a